The "Table of Relations" and Music Psychology in Hugo Riemann's Harmonic Theory

Michael Kevin Mooney

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UMI
300 North Zeeb Road
Ann Arbor, MI 48103

Abstract

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Hugo Riemann (1849–1919) belonged to an era of positivism during which musicology looked to the models and methods of the natural sciences. True to his time, Riemann attempted to establish a harmonic theory on acoustical and physiological grounds. Finding these grounds inadequate, he later construed tonal relations psychologically. Riemann's role in the history of music psychology is not well documented, probably because his contributions were scattered over an uncommonly prolific and varied publishing career.

In this study, we trace the emergence of music psychology in Riemann's harmonic writings, with reference to a construct that reappeared frequently in his work. This construct—the *Verwandschaftstabelle*, or Table of Relations—had a history prior to Riemann in the music theories of Leonhard Euler (1707–83) and Arthur von Oettingen (1836–1920). Riemann initially used the Table to summarize acoustic relations, but by the end of his career it had become a psychological hypothesis underlying a complex and idiosyncratic theory of harmony. We believe a study of the Table of Relations is corequisite to any full appraisal of Riemann's harmonic theory.

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I am grateful to Fred Lerdahl and Ian Bent for their careful readings of this work, and for their patience and encouragement over the course of its development.

The work is dedicated to my wife Marianne, for her love, support, and endurance.

Spiegel: noch nie hat manwissend beschrieben,
was ihr in euerem Wesen seid.

Rilke

Introduction

The Table of Relations is a two-dimensional array consisting of perfect fifths and major thirds. These intervals are "pure" (2 : 3 fifths, 4 : 5 thirds), and occur along interlocking horizontal and vertical axes that can be extended without limit. Pure intonation and extension distinguish the Table from equal-tempered constructs, such as the circle-of-fifths or Schoenberg's "Chart of the Regions," where octave equivalence can be invoked to set boundaries. The Table has no boundaries. It is appropriate to think of this vast tonal network spatially; it is a space whose elements are pitches, not pitch classes, and are thus distinct both in location and identity. Each pitch "carries ID," as it were—a number representing its acoustical frequency—and coordinate pairs, serving as addresses, could be issued for every pitch in the space.

The Table has received little attention on this side of the Atlantic; however, two German scholars have discussed it at length. Martin Vogel has outlined its early history in "Die Musikschriften Leonhard Eulers" (1960), an essay written to commemorate the 250th birthday of the famous Swiss mathematician. The Table has been prominent in subsequent work by Vogel, but has served mainly to forward his own theoretical views. Renate Imig's highly informative *Systeme der Funktionsbezeichnung in den Harmonielehre seit Hugo Riemann* (1970), has linked the Table with various harmonic systems since Riemann, but has not thoroughly investigated its significance to Riemann's harmonic theory. Imig does relate the Table to Riemann's function symbols (a relation that is only implicit in Riemann's own writings), but does not probe its relation to Riemann's music psychology, particularly his notion that listeners construct mental representations or *Tonvorstellungen* of what they hear. The present study is the first to treat the Table in connection with Riemann's psychological conception of tonality.

Overview

Chapter 1 of our study introduces the Table in connection with Euler's music theory. Riemann opposed Euler's numerical approach—especially as this pertained to issues of consonance and dissonance—but by adopting the Table of Relations instead of a circle-of-fifths model, he placed himself within a speculative tradition that included Euler and excluded mainstream theorists such as Rameau, Mattheson, and Heinichen.

Chapter 2 presents Oettingen's theory of harmonic dualism. Oettingen's resurrection of the Table as a model for harmonic dualism is of singular importance, since the dualist perspective—exemplified so well by the Table's symmetries—became a hallmark of Riemann's own theory. The Euler connection we suggest in Chapter 1 was largely a by-product of Oettingen's substantial and direct influence on Riemann.

Chapter 3 is an extended treatment of harmonic function, the other hallmark of Riemann's harmonic theory. We develop the notions of "categorical" and "chordal" function in this chapter—the first an abstract view inspired by Hauptmann, the second a more concrete view inspired by Oettingen—and relate these to two distinct paradigms: the *große Cadenz*, and the Table of Relations. Our thesis is that as Riemann shifted away from the categorical view and toward the chordal view, the Table came to dominate his harmonic conception.

Chapter 4 compares Riemann's view of music perception with that of Helmholtz. The *Vorstellung* concept (introduced in Chapter 3) is addressed in a broader historical context, and opposed to Helmholtz's empirical notion of *Tonempfindung*. The dichotomy between mental and theoretical representations, *Vorstellungen* and *Darstellungen*, is taken up in the course of a general discussion of tone psychology and music psychology.

Chapter 5 presents Riemann's system of harmonic *Schritte* and *Wechsel*. We first clarify pertinent aspects of Riemann's *Klangschlüssel* (chord notation) and *Funktionbezeichnung* (function notation). Then we explore some group-theoretic properties of Riemann's *Schritte* and *Wechsel*, and formalize the notion of an SW-system. A modified version of the Table, possessing six transformational axes, is presented toward the end of the chapter.

Notational Conventions

Two notational conventions will be encountered throughout this study: 1) a letter notation, or *Buchstabentonschrift*, is used to represent pitches on the Table; 2) intonation differences between pitches sharing the same *Buchstabe* are shown with horizontal lines, or *Striche*. In general, a line below a letter name represents a pitch one syntonic comma (about 21.5 cents) lower than its unlined counterpart. A line above a letter name represents the opposite. Tuning discrepancies are compounded as one moves further away from the Table's central series, and additional *Striche* are needed. Two lines below a letter name represent a pitch two commas lower than its unlined counterpart, three lines three commas, and so on. The combination of *Buchstaben* and *Striche* allows one to make precise distinctions between pitches such as 'e̲' ("third" of C), 'e' ("fourth fifth" of C), and 'e̲' ("four fifths below, two thirds above" C). We shall see that this *Buchstabe-Strich* notation developed erratically; the various modifications it underwent will be addressed as the need arises (see Chap. 3, n. 62).

Buchstaben combined with *Striche* are adequate for expressing intonation, but give only a general sense of the location of pitches on the Table: 'e' is on the central series, 'e' is on the series one step "north" of center, 'e' is on the series two steps "north" of center, and so on. In general, the number of *Striche* gives the distance "north" or "south" of center, but not the exact location within a series. To pinpoint location, we shall sometimes specify the number of fifths and/or thirds between central 'c' (our point of reference) and the pitch in question. In particular, we shall use Riemann's symbols Q (*Quint*) and T (*Terz*) for upper fifths and thirds, and their inverses -Q and -T for lower fifths and thirds: The pitch 'e' is therefore T of 'c', 'e̲' is 4Q of 'c', and 'e̲' is -4Q + 2T of 'c'. We shall use the same notation to describe the relative location of pitches: The distance from 'e' to 'e' is -T + 4Q.

When referring to specific pitch in this study we shall use the system adopted by the Acoustical Society of America: Cello C is C2, viola C is C3, middle C is C4, and so on.

CHAPTER 1: LEONHARD EULER AND THE "MIRROR OF MUSIC"

1.1 Introduction

The story of the Table of Relations begins with the eminent Swiss mathematician Leonhard Euler (1707–83), one full century before its appearance in Riemann's dissertation "Ueber das musikalische Hören" (1873). Euler began his career at the St. Petersburg Imperial Academy as an instructor in physics and mathematics, but was appointed director of the "mathematische Klasse" at the Berlin Academy in 1741. He held this post for twenty-five years before returning to St. Petersburg in 1766 where he began work on "De harmoniae veris principiis per speculum musicum repraesentatis," the last of five music-theoretical treatises written over the course of an amazingly prolific career.[1] The treatise was completed in 1773—by which time Euler was completely blind—and presented to the Academy on the 22nd of March that same year. Near the end of this work, Euler arranged the twelve tones of his diatonic-chromatic scale in a tabular format so that perfect fifths lay in the horizontals and major thirds in the verticals. He called this arrangement of tones *speculum musicum*, or the

[1] These works in chronological order are: "Musices theoreticae systema," (unpublished notebook, ca. 1726); *Tentamen novae theoriae musicae ex certissimus harmoniae principiis dilucide expositae* (St. Petersburg: ex typographia Academiae scientiarum, 1739); "Conjecture sur la raison de quelques dissonances généralement reçues dans la musique," *Mémoires de l'academie des sciences de Berlin* 20 (1764): 165–73; "Du véritable caractère de la musique moderne," *Mémoires de l'academie des sciences de Berlin* 20 (1764): 174–99; "De harmoniae veris principiis per speculum musicum repraesentatis," *Novi commentarii academiae scientiarium Petropolitanae* 18 (1773): 330–53. Contributions to acoustics include: *Dissertatio physica de sono* (Basel: typis E. & J. R. Thurnisiorum, 1727); *De la propagation du son* (1759); *Lettres à une princesse d'Allemagne sur divers sujets de physique et de philosophie* [1760–62] (St. Petersburg: Imprimerie de l'Académie impériale des sciences, 1768–72); *Eclaircissemens plus détaillés sur la génération et la propagation du son et sur la formation de l'écho* (1765).

"mirror of music." Euler's mirror is shown in Example 1-1, as it appears in the modern *Leonhardi Euleri Opera omnia*.[2]

EXAMPLE 1-1: EULER'S "MIRROR OF MUSIC"

F — C — G — D
| | | |
A — E — H — F*s*
| | | |
C*s* — G*s* — D*s* — B.

In this chapter we shall treat Euler's music theory in broad terms, to the extent that is needed for an understanding of the mirror of music. Euler was not a major influence on Riemann and it is worth stating at the outset that the mirror and Table of Relations differ in significant respects. Because the mirror reflects a particular division of the octave—Euler's *genus diatonicum-chromaticum* (more on this below)—each of its elements appears just once along the horizontal and vertical axes; Horizontal *Striche* are unnecessary, since no two pitches in this model share the same letter name. A kind of network does arise from Euler's arrangement of the twelve tones, but one must distinguish this from Riemann's Table, which is open-ended (not confined to a single division of the octave), and represents pitch, chord, and key relations. Martin Vogel (1960) has exaggerated the line of continuity between these models by presenting an embellished version of the mirror, which contains letter name redundancies and corresponding *Striche*. There are points of

[2] Leonhard Euler, "De harmoniae veris principiis per speculum musicum repraesentatis," in *Leonhardi Euleri Opera omnia sub auspiciis Societatis scientiarum naturalium Helveticae*, ed. A. Speiser, ser. 3, no. 11 (Zürich: Orell Füssli, 1960), 584. Further references to Euler's work cite the modern *Opera omnia*, and give original chapter and sectional numbers where applicable.

contact between Riemann and Euler (see 3.10), but Riemann's predecessor as far as the Table was concerned was Arthur von Oettingen. It is true that the "arrangement of tones called *speculum musicum* by Euler was discovered anew by Oettingen and gained acceptance in music acoustics through Helmholtz," but one must be wary of attributing music-theoretical influence on this basis alone.[3]

Early in his career, Riemann consigned Euler to a tradition that had tried to establish music theory on mathematical foundations. Euler was one in a long line of *Kanoniker* extending back to Pythagoras, who had engaged in the hapless pursuit of relating tone to number. Abstract laws of number had little to do with music in Riemann's opinion, and the *Kanoniker*'s obsession with number had hindered the progress of music theory by obscuring the importance of human perception. Riemann traced his own lineage to Aristoxenus and used the word *Harmoniker* to designate an opposing tradition, which eschewed the reliance upon numbers and arcane formulae.[4] For Riemann, hearing was the ultimate foundation of music theory—music was a manifestation of innate psychological processes, and the evidence of perception was thus of vital

[3] Martin Vogel, "Die Musikschriften Leonhard Eulers," in *Leonhardi Euleri Opera omnia*, ser. 3, no. 11: lx. Vogel's misrepresentation of the mirror is doubly unfortunate since it appears in the collected edition of Euler's work and in one of very few extended commentaries on Euler's music theory. Euler used *Striche* to indicate octave relations, a convention in acoustic treatises of the eighteenth and nineteenth centuries but one at variance with their usage to indicate tuning discrepancies.

[4] Hugo Riemann, "Ueber das musikalische Hören," (Leipzig: F. Andrä, 1874), 1–2. Riemann writes that Pythagoreans "sought the essence of musical consonance in the simplicity of proportion, formed by the string lengths of two tones" [suchte das Wesen musikalischer Consonanz in Einfachheit der Proportion, welche die verschiedenen Saitenlängen zweier Töne bildeten], whereas Aristoxenians believed "Numbers had nothing to do with consonance and dissonance and only the ear was judge" [Zahlen hätten mit Consonanz und Dissonanz nichts zu thun und nur das Ohr sei Richter]. Both branches of theory were alive and well in Riemann's day: "It is still like this today; the two opposing views remain as they were in the time of the *Harmoniker* (Aristoxenians) and *Kanoniker* (Pythagoreans)." [So ist es noch heute; die beiden einander gegenüberstehenden Ansichten sind heute dieselben wie zur Zeit der Harmoniker (Aristoxener) und Kanoniker (Pythagoräer).]

importance to music theory. Riemann considered himself the most recent in a line of distinguished *Harmoniker*, which included Rameau, Hauptmann, Oettingen, and Helmholtz. His readiness to exalt the evidence of the senses was typical of an age that sought to assimilate the methods of natural science to the study of human psychology.[5] Although his predecessors were more cautious in their appraisal of "psychological facts," none at least considered number the bedrock of music theory. This single fact was of supreme importance for Riemann; it divided the entire history of music theory—*Kanoniker* on one hand, *Harmoniker* on the other. From the beginning, Riemann and Euler were at opposite ends of the music-theoretical spectrum.

Euler's music theory is nevertheless an integral part of the Table's history. The reputation of the mathematician lent a general authority to his musical ideas, and his work in acoustics set an agenda for theorists and scientists in the nineteenth century. Riemann would not have pressed the distinction between consonance and dissonance so vigorously had Euler's theory—which admitted of degrees of consonance but gave no firm definition of dissonance—gone unnoticed. Euler's work thus places Riemann in context, and illuminates in particular his tangled relations with Oettingen and Helmholtz, who were both more generous in their assessment of Euler. Helmholtz believed that Euler's theory of consonance had "proved itself well," since its results agreed in many cases

[5] See Scott Burnham, "Method and Motivation in Hugo Riemann's History of Harmonic Theory," *Music Theory Spectrum* 14/1 (1992): 2. Riemann rejected acoustics and mathematics as foundations for music and turned increasingly to psychology. In doing so, Burnham says, he appeared "to replace one set of a priori suppositions with another, distinctly less verifiable, set. Yet this sort of transaction was quite simply the privilege of his age. Just as naturalist authors, especially in Germany, sought to demonstrate that human psychology was a determinable affair, subject to the indifferent control of natural laws, so too did thinkers like Riemann place confidence in the hard validity and causal consistency of so-called 'psychological facts.'"

with his own.[6] Had Riemann been less intractable, he might even have seen a parallel between Euler's *Substitutionslehre*—the idea that simple harmonic ratios can stand for complex ones—and his own conviction that listeners prefer the simplest *Tonvorstellungen*, or mental representations of sound (see 3.9 and 4.4). Both the *Substitutionslehre* and the doctrine of *Tonvorstellung* involved a quasi-Kantian distinction of the perceiving mind from the things perceived.

1.2 Pitch Space: Distance and Derivation

The mirror of music is a pitch space in the slimmest sense of the term. It does not address chord or key relations, and treats pitch abstractly rather than perceptually; Euler uses the mirror to show how pitches are derived from the prime numbers 2, 3, and 5, but does not inquire into the perceived distances among these pitches. Riemann also emphasizes derivation—though of a different sort—in his use of the Table. As his theory of harmonic functions evolved, chord derivation was greatly elaborated until there was a direct means of transformation between any two major or minor chords.[7] Still, the Table was conceived within a

[6] Hermann von Helmholtz, *Die Lehre von den Tonempfindungen als physiologische Grundlage für die Theorie der Musik*, 6th ed. (Braunschweig: Fr. Vieweg und Sohn, 1913; reprint, Hildesheim: Georg Olms, 1968), 377 n. 1. Helmholtz writes: "I shall use the principle upon which Euler determined the degree of consonance of intervals and chords, because its consequences prove themselves well if one disregards combination tones." [Ich will das Prinzip, nach welchem Euler die Stufenzahlen von Intervallen und Accorden bestimmt, hierhersetzen, weil es in der Tat in seinen Konsequenzen, soweit nicht Kombinationstöne in Betracht kommen, sich gut bewährt.]

[7] Lewin's claim that Riemann did not appreciate the transformational character of his theory is debatable in light of Riemann's system of *Schritte* and *Wechsel* (see Chap. 5); David Lewin, *Generalized Musical Intervals and Transformations*, (New Haven: Yale Univ. Press, 1987), 177.

harmonic framework where it is conventional to speak of perceived distance as well as chord derivation.

Euler introduced symbols ($\pm$III and $\pm$V) to denote distances along the two axes, but the mirror mainly reflected a set of numerical relations: Its twelve tones constituted a scale, or genus, derived from a combination of prime numbers 2, 3, and 5. Such combinations symbolize frequency ratios in Euler's theory, and can describe various divisions of the octave; the series 2 : 3 : 4, for example, describes a three-note scale that divides the octave at the fifth. By taking the least common multiple (LCM) of these terms and factoring the result, one gets: 12 (LCM) = $2^2 \cdot 3^1$. Factored expressions such as $2^2 \cdot 3^1$ identify specific musical genera, and are termed *exponens* by Euler.[8] An *exponens* is a kind of numerical signature for a genus, and different genera naturally will have different *exponens*. The *exponens* of Euler's diatonic-chromatic genus is $2^n \cdot 3^3 \cdot 5^2$ (n ranges over the positive integers, so as to bring all twelve pitches into the same octave). We shall discuss this important genus at greater length below.

The prime numbers 2, 3, 5, and (eventually) 7 were the pillars upon which Euler constructed a system of eighteen musical genera. Because the mirror of music appeared in his final treatise, and with little explanation there, it is necessary to backtrack and discuss some general features of the theory that spawned this model. We shall discuss Euler's theory of consonance, his formation of musical genera, and his concept of substitution—in roughly that order—with the main discussion of the mirror occurring in the section on tonal genera. Euler's ideas about consonance and substitution were spurned by most eighteenth-century

[8] The expression $2^2 \cdot 3$ or, more generally, $2^n \cdot 3$ is *exponens* for the second of eighteen genera in Euler's system.

theorists, but fared relatively well in the "tone-psychological" climate of the later nineteenth century.

1.3 Musical Consonance and Simple Ratios

Euler's theory is founded on the assumption that consonant intervals are based on simple frequency ratios. The greater the consonance, the simpler the ratio. Euler claimed that "the more easily we perceive order in a thing, the simpler and more perfect we find it, and we are thus pleased. If the order is recognized with difficulty, and appears less simply and clearly, we register it with a certain grief."[9] A simple relation consisted of small whole-number ratios; the smaller the numbers, the clearer and more pleasant the psychological effect. We need not look as far back as Pythagoras to find theorists using ratios to represent intervallic consonance; several of Euler's more recent predecessors had built hierarchies of consonance on this principle too. According to Vogel, Euler found the germ of this idea in Descartes' statement that an interval "is perceived more easily by the senses when the difference of the parts is smaller. We may say that the parts of a whole object are less dissimilar when there is greater proportion between them."[10] Euler's more immediate influences, however, were Galilei, Saveur, and Leibniz.

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[9] Leonhard Euler, *Tentamen novae theoriae musicae,* chap. 2, §13, in *Leonhardi Euleri Opera omnia*, ed. E. Bernoulli, R. Bernoulli, F. Rudio, and A. Speiser, ser. 3, no. 1 (Leipzig and Berlin: B. G. Teubner, 1926). [Quo facilius ordinem, qui in re proposita inest, percipimus, eo simpliciorem ac perfectiorem eum existimamus ideoque gaudio et laetitia quadam afficimur. Contra vero si ordo difficulter cognoscatur isque minus simplex minusque planus videatur, cum quadam quasi tristitia eundem animadvertimus.] See also Charles S. Smith, "Leonhard Euler's *Tentamen novae theoriae musicae*: A Translation and Commentary" (Ph.D. diss., Indiana University, 1960), 71–72.
[10] René Descartes, *Compendium of Music,* trans. Walter Robert with an introduction and notes by Charles Kent, Musicological Studies and Documents, no. 8 (Rome: American Institute of Musicology, 1961), 17.

Leibniz's notion that the mind calculates intervallic consonance by counting the numerical terms of a proportion was of particular importance to Euler. Complex proportions were less amenable to this mental arithmetic, and Leibniz wrote in 1712 (to Christian Goldbach) that people were capable of counting only to five in music.[11] He believed that the number 7 might eventually be added to the first three primes, and Euler in his later work indeed counted 7 among the prime pillars of music.

The interest of eighteenth-century scientists in issues of consonance and dissonance was linked to fervent research into acoustics. Scientists had earlier believed all overtones to be harmonious, but had been without a means of calculating the fundamental and overtone frequencies of a sound. Initial attempts to measure the speed of sound were made by Mersenne in 1640, and by two Italian physicists—Giovanni Borelli and Vincenzo Viviani—in 1656.[12] Newton then undertook the problem in the eighteenth century, but the speed he calculated was too low and was not corrected until 1868. The study of acoustics nevertheless was able to proceed upon a stable foundation, for acousticians discovered that even if they could not measure pitch exactly they could determine the correct frequency ratio between a fundamental and its overtones by drawing on mechanical principles. Since frequency ratios determined intervals, the study of consonance and dissonance could be codified to some extent.

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[11] Gottfried Wilhelm Leibniz, *Epistolae ad diversos*, ed. C. Kortholt, vol. 1 (Leipzig: Bern. Christoph Brettkopf, 1734), 239. Leibniz writes: "In music we do not count beyond five; we are thus as those in the field of arithmetic who have not yet advanced beyond ternary forms, and in relation to whom the German phrase about the simple man is fitting: He can not count higher than three." [Nos in Musica non numeramus ultra quinque, similes illis populis, qui etiam in Arithmetica non ultra ternarium progrediebantur, et in quibus phrasis Germanorum de homine simplice locum haberet: Er kan nicht über drey zählen.]

[12] Harvey E. White and Donald H. White, *Physics and Music: The Science of Musical Sound* (Philadelphia: Saunders College, 1980), 39.

Euler's early publication *Dissertatio physica de sono* (1727), completed when he was just twenty years old, contributed to this codification and was highly influential among acousticians. In this and later music-related works, he expressed his findings with numerical ratios.

A quantitative measure of acoustical consonance was introduced in *Tentamen novae theoriae musicae* (1739), the magnum opus of Euler's music theory. Here, Euler used frequency ratios to calculate the *gradus suavitatis* or "degree of sweetness" of intervals and chords, and to determine the *exponens* of various tonal genera. The result was a systematic and thorough-going treatment of consonance and scale. Euler began with a general discussion of the principles of sound and hearing—in which he gave a surprisingly accurate formula for the frequency of a vibrating string[13]—and concluded with an elaborate theory of modulation that gauged the relative consonance of progressions to different keys. Euler's goal was to establish music theory upon a secure and scientifically exact foundation. He believed that number provided such a foundation, since harmony, melody, and rhythm were all expressible as numerical proportions.

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[13] Euler, *Tentamen*, chap. 1, §9. In Euler's formula $\frac{355}{113}\sqrt{\frac{3166n}{a}}$, *a* is the vibrating length of string, *n* is the ratio of suspended weight to the weight of the string—i.e. the string tension—3166 is the length of a seconds pendulum, and $^{355}/_{113}$ is the traditional approximation of π. String and pendulum lengths are in thousandths of a Rhenish foot (one Rhenish foot equals 12.356 inches). Euler had introduced the symbol π for the ratio of a circle's circumference to its diameter in 1737 but did not use it in *Tentamen*. According to Ellis, the results of Euler's formula "on account of the necessary thickness of the string… could not be trusted within 5 vib." See Hermann von Helmholtz, *On the Sensations of Tone as a Physiological Basis for the Theory of Music*, trans. A. J. Ellis (New York: Dover, 1954), 441.

1.4 Degree of Intervallic Consonance

Euler assumed that intervals could be ranked on a continuous scale of consonance, since consonance increased or decreased with the progressive differentiation of frequency ratios. He supposed that expanding or contracting an interval by an octave would change the level of consonance by one degree—with expansion resulting in a less consonant interval, and contraction in a more consonant interval—and used whole numbers to indicate the degree of consonance. The smaller the numbers, the more perfect the consonance: The intervals 1 : 1, 1 : 2, and 1 : 4 belonged respectively to the first, second, and third degrees of consonance, and any ratio of the form $1 : 2^n$ belonged to the degree (n + 1).[14] Euler's calculations for octave and octave compounds are summarized in Example 1-2.

EXAMPLE 1-2: *GRADUS SUAVITATIS* OF OCTAVES[15]

Interval	Degree of Consonance
1 : 1 (unison)	1
1 : 2 (octave)	2
1 : 4 (double octave)	3
1 : 8 (triple octave)	4
1 : 16 (quadruple octave)	5
$1 : 2^n$	(n +1)

[14] Euler, *Tentamen,* chap. 2, §23.
[15] Vogel, xlvii.

Euler next determined the degree of consonance for prime numbers. Prime numbers received a consonance rating equal to themselves, so that 1 : 3 had a rating of 3, 1 : 5 a rating of 5, and 1 : p, where p is prime, a rating of p. For octave contractions of prime ratios—1 : 3 to 2 : 3, for example—Euler took the value of p and added one rating point for each contraction. Since 1 : 3 belonged to the third degree of consonance, 2 : 3 belonged to the fourth degree, and $2^n : p$ belonged to the $(p + n)$ degree.[16] This method differs from the one used for octaves, where compound intervals received a higher *gradus suavitatis* (*GS*) rating than contracted ones. Euler's results corresponded to general intuitions of intervallic consonance, however, and these results—not the manner of achieving them—recommended his work to Helmholtz and various tone psychologists of the nineteenth and early twentieth century.[17] Examples 1-3a and 1-3b summarize Euler's *GS*-ratings for prime intervals.

EXAMPLE 1-3A: *GRADUS SUAVITATIS* OF PRIMES[18]

Prime Interval	Degree of Consonance
1 : 3 (12th)	3
1 : 5 (17th)	5
1 : 7 (21st)	7
1 : p	p

[16] Euler, *Tentamen*, chap. 2, §25.
[17] For example, Felix Krüger, "Die Theorie der Konsonanz," *Psychologische Studien* 4 (1909): 250; and Felix Auerbach, *Die Grundlagen der Musik* (Leipzig, 1911), 184.
[18] Vogel, xlvii; Helmholtz, 377.

EXAMPLE 1-3B: *GRADUS SUAVITATIS* OF PRIMES
(COMPOUND AND CONTRACTED)

Prime Interval	Degree of Consonance
1 : 5 (17th)	5
2 : 5 (10th)	6
4 : 5 (3rd)	7
$2^n : p$	$p + n$

Euler was ultimately able to specify the consonance of any interval 1 : *P*, where *P* is a positive integer, by factoring *P* and using the prime form to determine its *GS*-rating. Musically speaking, it was unnecessary to consider all but a handful of ratios. The degree of consonance for extremely large intervals, 1 : 360 for example, had no bearing on musical practice and was purely a matter of speculation. Euler nevertheless compiled a table of all intervals belonging to the first ten degrees of consonance.[19] This table of *consonantiarum bisonarum* (bisonant or intervallic consonances) is reproduced in Example 1-4. Notice that Gr. I, or the first degree of intervallic consonance, is omitted from the table. Euler explained that since the two sounds of a consonance must be different, the unison 1 : 1 is not technically a consonance, and the simplest consonance (1 : 2) must therefore belong to the second degree.

[19] Euler, *Tentamen,* chap. 4, §11.

EXAMPLE 1-4: TABLE OF *CONSONANTIARUM BISONARUM*

11. Hoc modo sequentem confeci tabulam consonantiarum bisonarum, in qua eae sunt secundum suavitatis gradus supra expositos dispositae, ad decimum usque gradum:

Gr. II:	**1 : 2.**
Gr. III:	**1 : 3, 1 : 4.**
Gr. IV:	**1 : 6, 2 : 3, 1 : 8.**
Gr. V:	**1 : 5, 1 : 9, 1 : 12, 3 : 4, 1 : 16.**
Gr. VI:	**1 : 10, 2 : 5, 1 : 18, 2 : 9, 1 : 24, 3 : 8, 1 : 32.**
Gr. VII:	**1 : 7, 1 : 15, 3 : 5, 1 : 20, 4 : 5, 1 : 27, 1 : 36, 4 : 9, 1 : 48, 3 : 16, 1 : 64.**
Gr. VIII:	**1 : 14, 2 : 7, 1 : 30, 2 : 15, 3 : 10, 5 : 6, 1 : 40, 5 : 8, 1 : 54, 2 : 27, 1 : 72, 8 : 9, 1 : 96, 3 : 32, 1 : 128.**
Gr. IX:	**1 : 21, 3 : 7, 1 : 25, 1 : 28, 4 : 7, 1 : 45, 5 : 9, 1 : 60, 3 : 20, 4 : 15, 5 : 12, 1 : 80, 5 : 16, 1 : 81, 1 : 108, 4 : 27, 1 : 144, 9 : 16, 1 : 192, 3 : 64, 1 : 256.**
Gr. X:	**1 : 42, 3 : 14, 6 : 7, 1 : 50, 2 : 25, 1 : 56, 7 : 8, 1 : 90, 2 : 45, 5 : 18, 9 : 10, 1 : 120, 3 : 40, 5 : 24, 8 : 15, 1 : 160, 5 : 32, 1 : 162, 2 : 81, 1 : 216, 8 : 27, 1 : 288, 9 : 32, 1 : 384, 3 : 128, 1 : 512.**

The fact that Euler bothered with such intervals as 1 : 384 or 1 : 512 (both degree 10) suggests that practical applications were of no special importance to him. At least as important were the extensions of his theory to complex harmonic phenomena of little or no musical use. To determine the degree of consonance for extremely large (1 : 360) or complex (80 : 81) ratios, Euler found the LCM of the terms and expressed this in prime form as we did for the series 2 : 3 : 4. Using the LCM's prime form, Euler then calculated a *GS*-rating by subtracting the number of terms from the sum of terms, and adding 1. He expressed this formula for determining the *GS*-rating of a given *exponens* as $s - n + 1$, where s is the sum of prime factors, and n is the number of prime factors.[20] For convenience's sake, he also provided an LCM table for the first sixteen degrees of consonance. If one did not want to go to the trouble of factoring 360, in the interval 1 : 360, one could simply scan the table until 360 was found among LCMs belonging to the twelfth degree of consonance. We reproduce Euler's LCM table in Example 1-5.[21]

EXAMPLE 1-5: EULER'S LCM TABLE

I.	1;
II.	2;
III.	3, 4;
IV.	6, 8;
V.	5, 9, 12, 16;
VI.	10, 18, 24, 32;
VII.	7, 15, 20, 27, 36, 48, 64;
VIII.	14, 30, 40, 54, 72, 96, 128;
IX.	21, 25, 28, 45, 60, 80, 81, 108, 144, 192, 256;
X.	42, 50, 56, 90, 120, 160, 162, 216, 288, 384, 512;
XI.	11, 35, 63, 75, 84, 100, 112, 135, 180, 240, 243, 320, 324, 432, 576, 768, 1024;
XII.	22, 70, 126, 150, 168, 200, 224, 270, 360, 480, 486, 640, 648, 864, 1152, 1536, 2048;
XIII.	13, 33, 44, 49, 105, 125, 140, 189, 225, 252, 300, 336, 400, 405, 448, 540, 720, 729, 960, 972, 1280, 1296, 1728, 2304, 3072, 4096;
XIV.	26, 66, 88, 98, 210, 250, 280, 378, 450, 504, 600, 672, 800, 810, 896, 1080, 1440, 1458, 1920, 1944, 2560, 2592, 3456, 4608, 6144, 8192;
XV.	39, 52, 55, 99, 132, 147, 175, 176, 196, 315, 375, 420, 500, 560, 567, 675, 756, 900, 1008, 1200, 1215, 1344, 1600, 1620, 1792, 2160, 2187, 2880, 2916, 3840, 3888, 5120, 5184, 6912, 9216, 12288, 16384;
XVI.	78, 104, 110, 198, 264, 294, 350, 352, 392, 630, 750, 840, 1000, 1120, 1134, 1350, 1512, 1800, 2016, 2400, 2430, 2688, 3200, 3240, 3584, 4320, 4374, 5760, 5832, 7680, 7776, 10240, 10368, 13824, 18432, 24576, 32768.

[20] Euler, *Tentamen*, chap. 4, §6.
[21] Euler, *Tentamen*, chap. 2, §31.

By writing 360 as $2^3 \cdot 3^2 \cdot 5^1$ and applying the formula $s - n + 1$, one obtains a *GS*-rating of 12. This is not as easy a process as consulting the LCM table, but it does not take too long to master. One will occasionally encounter LCMs that are unlisted in Euler's table, and will have to apply the formula in such cases. We apply Euler's formula to the ratio 80 : 81—that of the syntonic comma—in Example 1-6.

EXAMPLE 1-6: *GRADUS SUAVITATIS* OF THE SYNTONIC COMMA

1. Find the LCM of the ratio:
   a) write each term as $p_1a^1 \cdot p_2a^2 \cdot \ldots p_na^n$, where $p_1, p_2, \ldots p_n$ are prime factors and $a^1, a^2, \ldots a^n$ are whole number exponents:
   $$80 = 2^4 \cdot 5^1, 81 = 3^4$$
   b) LCM is the product of the factorization of each term:
   $$2^4 \cdot 3^4 \cdot 5^1 = 6400$$

2. Calculate degree of consonance (*GS*) for LCM:
   a) $GS(\text{LCM}) = s - n + 1$
   $$= (p_1a^1 + p_2a^2 + \ldots p_na^n) - (a^1 + a^2 + \ldots a^n - 1)$$
   b) $GS(6400) = (2)(4) + (3)(4) + (5)(1) - (4 + 4 + 1 - 1)$
   $$= 8 + 12 + 5 - 8$$
   $$= 17$$

Notice that if two numbers p and q are given, the *GS* of their product is the same as the sum of their *GS*s minus one. For example, where $p = 3$ and $q = 5$, $GS(3 \cdot 5) = GS(3) + GS(5) - 1$. The *GS*-rating in both cases is 7.[22]

[22] Euler, *Tentamen*, chap. 2, §25–26. Euler writes: "If the ratio 1 : *P* belongs to degree p, and the ratio 1 : *Q* to degree q, the ratio 1 : *PQ* will belong to the degree $p + q - 1$." [Si ratio 1 : *P* ad gradum p pertineat et ratio 1 : *Q* ad gradum q, pertinebit ob allatas rationes ratio 1 : *PQ* ad gradum $p + q - 1$.] See Hermann R. Busch, *Leonhard Eulers Beitrag zur Musiktheorie* (Regensburg: Gustav Bosse Verlag, 1970), 34.

1.5 Degree of Chordal Consonance

Euler determined the *gradus suavitatis* of triads and larger chords by using the same two steps as in Example 1-6: He first found the *exponens* of the chord, and then calculated the degree of consonance with the formula $s - n + 1$. Example 1-7 illustrates this process for the ratios representing the major triad and major-minor seventh chord.

EXAMPLE 1-7: *GRADUS SUAVITATIS* OF MAJOR AND MAJOR-MINOR SEVENTH CHORDS

1. Find the LCM of the ratios:

a)	major chord	=	4 : 5 : 6
	LCM	=	60 ($2^2 \cdot 3^1 \cdot 5^1$)
b)	major-minor 7th	=	4 : 5 : 6 : 7
	LCM	=	420 ($2^2 \cdot 3^1 \cdot 5^1 \cdot 7^1$)

2. Calculate degree of consonance (*GS*) for LCM:

a) $GS(60)$

$$= s - n + 1$$
$$= (p_1a^1 + p_2a^2 + \ldots p_na^n) - (a^1 + a^2 + \ldots a^n - 1)$$
$$= (2)(2) + (3)(1) + (5)(1) - (2 + 1 + 1 - 1)$$
$$= 4 + 3 + 5 - 3$$
$$= 9$$

a) $GS(420)$

$$= s - n + 1$$
$$= (p_1a^1 + p_2a^2 + \ldots p_na^n) - (a^1 + a^2 + \ldots a^n - 1)$$
$$= (2)(2) + (3)(1) + (5)(1) + (7)(1) - (2 + 1 + 1 + 1 - 1)$$
$$= 4 + 3 + 5 + 7 - 4$$
$$= 15$$

Since Euler based all of his calculations on positive integers, he simply reversed the terms of the major triad (4 : 5 : 6) to represent the minor triad

(6 : 5 : 4), and the *GS*-rating remained the same for both harmonies.[23] In the case of the major-minor seventh chord, we shall see that 4 : 5 : 6 : 7 was actually a simplified substitute for a more complex ratio.

1.6 New Solutions, New Problems

The theoretical practice of relating consonance to number took various forms throughout history. The Pythagoreans had originally devised a method in which the number 1 was subtracted from both terms of a ratio, and the sum of the two remainders was taken for the degree of consonance.[24] Through this method, the octave 1 : 2 received a consonance rating of 1, since (1 – 1) : (2 – 1) = 0 + 1 = 1, whereas it received a rating of 2 in Euler's system. In both systems, the twelfth 1 : 3 was more consonant than the fifth 2 : 3, but for Pythagoreans the *GS*-ratings were 2 and 3 whereas for Euler they were 3 and 4. The chief difference between the two methods lay in Euler's use of prime forms to calculate intervallic and chordal consonance. This innovation quickly pointed up the limitations the earlier approach. For example, the eleventh 3 : 8 received a *GS*-rating of 9 (2 + 7) in the Pythagorean system and a rating of 6 in Euler's

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[23] The Swiss scientist Jean-Adam Serre extended Euler's system by using the reciprocal terms of the harmonic series: Jean-Adam Serre, *Essais sur les principes de l'harmonie* (Paris, 1753; reprint, New York: Broude, 1967), 135. Serre writes: "M. Euler in his calculation of the numerical relation between musical sounds has neglected their inverses, that is, those terms forming the harmonic or fractional series, 1/2. 1/3. 1/4. 1/5. 1/7., & c." [M. Euler dans son Calcul des Rapports numériques qui ont lieu entre les Sons musicaux... a négligé leurs Inverses, c'est-à-dire, ceux qui forment la Progression harmonique ou fractionnaire, 1/2. 1/3. 1/4. 1/5. 1/7., & c.] Serre uses a Table-like array at the end of this work to show the symmetrical relation between the dominant seventh (*dominante prédomine*) and supertonic seventh (*soudominant prédomine*).

[24] See Andrew Barker, ed., *Greek Musical Writings: II. Harmony and Acoustic Theory* (Cambridge: Cambridge Univ. Press, 1989), 34–35.

system. Euler's rating conformed better with contemporary practice, and to the physical consonance of the interval.[25]

The role of prime numbers in Euler's theory is analogous to the role played by tonic, dominant, and subdominant in Rameau's theory. Euler found that the music theory of his contemporaries favored intervals that could be expressed through the prime numbers 2 (octave), 3 (fifth), and 5 (third). He excluded primes larger than 7 because of their high *GS*-ratings; intervals such as 1 : 11 (*GS* = 11) or 1 : 13 (*GS* = 13) went beyond the pale of an already extensive catalogue of intervallic consonance (see Example 1-3). The number 7 played a changeable role in Euler's theory, but was initially less important than the smaller primes. Euler shared the indecision of his contemporaries on the status of the minor seventh. At first he believed (with Leibniz) that the mind counted no further than 5 in music, and included no genera with 7 in their *exponens*. This, however, did not exclude intervals such as 1 : 7 or 4 : 7 from consideration. On the contrary, 4 : 7 (perfect fourth) received a consonance rating of 9, whereas 4 : 5 (major third) received an only slightly lower rating of 7. Vogel, who has written extensively on the music-theoretical status of the number 7, emphasizes Euler's ultimate inclusion of the number 7 and regards his own theory as stemming from Euler's in this respect. There can be no question of Euler's acceptance of the seventh in later work. In "Conjecture sur la raison de quelques dissonances généralement reçues dans la musique" (1764), he points to a "nouveau genre de musique" for which the numbers 2, 3, and 5

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[25] The Pythagorean and Eulerian rankings diverge sharply in the case of superparticular ratios. In the Pythagorean system, *GS*-ratings rise continuously as the terms of the ratios increase: For example, the ratings for 7 : 8, 8 : 9, 9 : 10, 10 : 11, 11 : 12, and 12 : 13 are 13, 15, 17, 19, 21, and 23, respectively. The ratings for the same ratios in Euler's system are 10, 8, 10, 16, 15, and 17. Euler claims that the Pythagorean "criteria for separating consonance from dissonance were not based on the nature of the matter, but deduced from questionable principles." Euler, *Tentamen*, chap. 4, §16; see also Busch, 41.

are no longer sufficient,[26] and in "Du véritable caractère de la musique moderne" (1764) he proposes a 24-note division of the octave where twelve *Tons étrangers* are obtained by multiplying the original twelve *Tons principaux* by seven.[27] Vogel nevertheless exaggerates the role of the minor seventh in Euler's theory. It played no role in the system of tonal genera, and therefore had no pertinence to the mirror of music (it was also alien to the Tables of Oettingen and Riemann). Vogel stands practically alone among *Kanoniker* in assigning equal weight to the prime numbers 2, 3, 5, and 7.[28]

Because Euler measured acoustical consonance on a continuous scale, he made no strict distinction between consonance and dissonance. Instead, in "Du véritable caractère," he maintained that sounds were more or less simple according to his system of numerical rating. This gradational notion of consonance was later refined by Helmholtz, and was popular among nineteenth-century scientists even if they could not always agree with Euler's results. Many of Euler's calculations from the least common multiple indeed resulted in questionable or outrightly absurd *GS*-ratings. This was especially evident when the principle was applied mechanically to octaves: $GS(1) = 1$, $GS(2) = 2$, $GS(3) = 3$, and so forth. It was a sign of the theory's weakness that an interval as large as 1 : 4 (triple octave) should have the same rating as the perfect fifth 2 : 3 ($GS = 4$).

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[26] Leonhard Euler, "Conjecture sur la raison de quelques dissonances généralement reçues dans la musique," in *Leonhardi Euleri Opera omnia*, ser. 3, no. 1:515, §16.
[27] Leonhard Euler, "Du véritable caractère de la musique moderen," in *Leonhardi Euleri Opera omnia*, ser. 3, no. 1:539, §44.
[28] The Dutch composer Adriaan D. Fokker has explored genera that include 7 in their *exponens*. See Adrian D. Fokker and Jan von Dijk, "Expériences musicales avec les genres musicaux de Leonhard Euler contenant la septième harmonique," in *Internationale Gesellschaft für Musikwissenschaft, Kongressbericht* (Basel: Barenreiter,1949),113–15.

There were similar problems in finding 4 : 5, 1 : 7, and 1 : 15 all equally consonant ($GS = 7$).

The problems were thorniest with chordal sonorities. Because the series 8 : 10 : 12 : 15 (major-seventh chord) and 4 : 5 : 6 : 8 (major triad with doubled root) shared the same LCM (120), the GS-rating for both sonorities was 10 (versus 9 for a simple major triad). If the frequency of one of these tones deviated only slightly—say 401 : 500 instead of 4 : 5 in the major triad—the GS-rating would rise spectacularly. Vogel admits that Euler's determination of chordal consonance is of limited use, and cites three examples that point up these limitations. In his first example, Vogel shows that the GS-rating of the augmented triad 16 : 20 : 25 is determined by the outer interval (16 : 25) alone, since its prime form ($2^4 \cdot 5^2$) contains that of the middle term ($2^2 \cdot 5^1$). Both the framing interval and the complete triad therefore belong to the faraway thirteenth degree, even though the interval usually sounds less dissonant than the chord. Vogel next cites the major seventh 8 : 15 as a framing interval, and finds that $GS = 10$ regardless of whether one adds a third, a fifth, or both to this interval. Indeed, the degree of consonance is not changed by adding the lower octaves 1 : 2 : 4 : 6. Vogel's last example, involving a major-minor seventh chord, is his most compelling. In this example, he shows how the seventh chord 36 : 45 : 54 : 64 and diatonic scale 36 : 40 : 45 : 48 : 54 : 60 : 64 receive the same GS-rating. The LCM of 36 : 45 : 54 : 64 is 8640, or $2^6 \cdot 3^3 \cdot 5^1$; since this also contains the factors of 40 : 48 : 60, $GS = 17$ for both collections.[29]

Euler was aware of such anomalies and tried to smooth them out in later treatises. In "Conjecture," for example, he said that a disparity always exists between the actual consonance of sounds and the way the sounds

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[29] Vogel, li–lii.

are represented psychologically. This dualism between objective and subjective—or psychological—modes of reality was the basis of an interesting theory of perception based on substituting one mode for the other. We shall say more about substitution at the end of this chapter. This, however, was just one explanation that Euler used to bolster his consonance theory. As early as *Tentamen,* he introduced the notions of "incomplete" and "complete" consonance to account for chords whose *GS*-ratings were unaffected through the addition of new tones.[30] The sonorities 1 : 2 : 3 and 1 : 2 : 3 : 6 both had an LCM of 6 and a *GS*-rating of 4, but the first was incomplete because it omitted one of the terms included in the *exponens* $2^1 \cdot 3^1$, namely 6. Euler's catalogue of chordal sonorities extended through the first twelve degrees of consonance—he went beyond the tenth degree because chords tended to be less consonant than intervals—but the concepts of incomplete and complete consonance allowed him to reduce many chords with the same *GS*-rating to a few basic types.

1.7 Reception of Euler's Theory of Consonance

Euler's conception of harmony was atomistic rather than holistic; that is, he considered chords and intervals as nothing more than the sum of their parts. Vogel notes that chordal inversion and doubling played no role in Euler's determination of consonance, which was a flaw, in his opinion, since a theory of consonance must begin with the fact that chords are perceived qualitatively as wholes and not quantitatively as frequency ratios. Like the Gestalt psychologists before him, Vogel believes that

[30] Euler, *Tentamen*, chap. 4, §21.

qualitative experience cannot be expressed quantitatively and that Euler's attempt at a graduated scale of consonance is therefore misguided.

Still, Vogel believes that Euler's results have musical worth. He attaches particular importance to the notion of octave gradation, which, in spite of some misgivings, he believes has practical value for composition.[31] He believes also that the prime numbers 2, 3, 5, and 7 provide a sounder foundation for tonal theory than the overtone series. From Rameau onward, music theorists have singled out the first five partials of this series as a natural foundation for music. Vogel believes that such reasoning is based on two assumptions that simplify the music-acoustical facts: 1) the assumption that one hears only the first five partials along with the fundamental; 2) the assumption of octave equivalence.[32] Euler's system distinguishes between octaves and provides a rationale for the natural seventh—two features that are central to Vogel's own work. It is worth noting that Rameau's "naturalist" foundation came under fire in its own day, when the eighteenth-century physicist Daniel Bernoulli established that the overtones of most natural bodies were actually dissonant. Those selected for music-making were unusual and chosen precisely because their overtones were pleasing to the ear. Bernoulli's work did not affect Euler's equating of simplicity with consonance. However, it did undermine the idea that simple ratios were consonant because simplicity itself was favored by nature.

Contemporaries had mixed, though mostly negative reactions to Euler's music theory. Johann Mattheson—never one to hold back—railed

[31] Vogel writes (liii): "Euler's interval table can serve as an excellent guide to the most consonant spacing of the two tones of an interval." [Eulers Zweiklangtabelle kann als vorzügliche Anleitung gelten, bei welchem Oktavabstand die beiden Töne eines Intervalls ihre höchste Konsonanz erreicht haben.]
[32] Vogel, liv.

against *Tentamen* and all "versunken Algebraisten [lost algebraists], Verhältnisfechter [champions for the cause of ratios], schulfächsische Proportionsleiter [proportion-pedants] und Rations-Händler [ratio-merchants]"[33] Lorenz Mizler had more in common with Euler, but objected to the use of prime numbers higher than 5. In his view, Euler's *Stufentabelle* "belongs in another world, since everything is put together according to different relations than in our world."[34] Even Nicolaus Fuß, Euler's protégé and earliest biographer, was reserved in his estimation of *Tentamen*, saying that it "had made no special impression, perhaps because it contained too much mathematics for musicians and too much music for mathematicians."[35] Rameau of course objected to Euler's

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[33] Johann Mattheson, "Die neue Zahl-Theorie," *Plus ultra, ein Stückwerk von neuer und mancherley Art*, 4 vols. (Hamburg: Johann Adolph Martini, 1754–56).

[34] Lorenz Mizler was editor of *Neue eröffnete musikalische Bibliothek*, the monthly periodical of the Korrespondierenden Sozietät der Musikalischen Wissenschaften (whose members included J. S. Bach and Telemann). This facinating periodical, issued between 1736 and 1754, treated musical issues from an eclectic range of mathematical and scientific perspectives. Mizler's annotated Latin-German translation of chaps. 1–4 of *Tentamen* appeared in vols. 3/1–3 (1746–47): 61–136, 305–46, 539–58; and 4/1 (1754): 69–103. Concerning Euler's use of prime numbers, Mizler wrote (vol. 3/2: 328): "For in all of our music, there are just three intervals arising directly from prime numbers: 1 : 2, 1 : 3, the compound fifth, and 1 : 5, the compound third; the ratio 1 : 15, which is the compound seventh, arises when one multiplies 3 and 5. Now, if I may say so, Herr Euler wants too much to prove for the advancement of musical science that there are more prime ratios in our music than those we have stated. If he will only show this to be so, I will consider him for the Apollo in music, but meanwhile I can only believe that his *Stufentabelle* belongs in another world, since everything is put together according to different relations than in our world." [Denn in unserer ganzen Musik haben wir nicht mehr als drey Intervalle, die aus Primzahlen bestehen, nämlich 1 : 2, 1 : 3, die erste zussamengesetzte Quinte, und 1 : 5, die zweyte zussamengesetzte grosse Terz, und die Verhältniß 1 : 15, so entsteht, wenn man 3 und 5 mit einander multiplicirt, und die dritte zusammengesetzte grosse Septime ist. Nun bitte Herrn Eulern gar sehr, er möchte doch zur Beförderung der musikalischen Wissenschaften darthun, daß noch mehr Verhältnisse in unserer Musik von Primzahlen sind, außer den angegeben; Wo er nur die Möglichkeit erweisen wird, so will ich ihn für den Apollo in der Musik halten, derweilen aber kann ich nicht anders glauben, als daß seine Stufentabelle in eine andere Welt gehört, da alles ganz nach andern Verhältnissen zusammengesetzt ist, als in unserer Welt.]

[35] Nicolaus Fuß, "Lobrede auf Herrn Euler...an 23 Oktober vorgelesen," in *Leonhardi Euleri Opera omnia*, ser. 1, no. 1 (Leipzig and Berlin: B. G. Teubner, 1911), lix. [...hat indessen kein sonderlich Aufsehn gemacht: vielleicht nur deswegen, weil es zuviel Mathematik für den Tonkunstler, und zuviel Musik für den Mathematiker enthält.]

acceptance of the seventh partial, as well as to the idea of octave gradation, but from their correspondence it is clear that each respected the other's viewpoint.[36] The Berlin composer-theorist Kirnberger was alone among eighteenth-century musicians in endorsing Euler's ideas, and accepting the natural seventh as a consonance. Kirnberger even introduced a special symbol *i* to denote the natural seventh.[37]

Riemann and Stumpf were among the nineteenth-century critics of Euler's theory of consonance. Riemann spoke of its "most baroque consequences,"[38] saying that *Tentamen* was "a warning for all times"[39] and "evidence that mathematics does not suffice for the foundation of a musical system."[40] Stumpf wrote that "nowhere does it show itself better than in the musical writings of the great mathematician, into what shoreless sea this manner of contemplation leads."[41] Finally, the German aesthetics historian Rudolf Schäfke mentions Euler dispassionately as a latecomer or "straggler of a past epoche."[42] His mathematical approach

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[36] Two letters, both written in 1752, survive what Jacobi calls a "doubtless prolific correspondence" between Rameau and Euler. See Erwin R. Jacobi ed., *The Complete Theoretical Writings of Jean-Philippe Rameau*, vol. 5 (Publications of the American Institute of Musicology: Miscellanea, 1969), xxxii–xxxiii; 146–48.

[37] The symbol *i* is more universally known as Euler's notation for the imaginary unit $\sqrt{-1}$ in algebra. Euler introduced this notation in "De insigni usu calculi imaginariorum in calculo integrali," *Nova Acta Academiae Petropolitanae* (1777): 3. Kirnberger used the symbol at least a decade earlier in his *Clavierübungen* (Berlin, 1761–63), and again in *Die Kunst des reinen Satzes in der Musik* (Berlin and Königsberg, 1776–79; reprint, Hildesheim: Georg Olms, 1968), 4. It is unclear whether Euler knew of Kirnberger's use of *i*; both were in Berlin until 1766, and Euler's music treatises from the 1760s focused on the issue of the natural seventh.

[38] Hugo Riemann, *Musikalische Syntaxis: Grundriß einer harmonischen Satzbildungslehre* (Leipzig: Breitkopf und Härtel, 1877), 10.

[39] Hugo Riemann, *Grundriß der Musikwissenschaft*, rev. ed. (Leipzig: Breitkopf und Härtel, 1918), 60.

[40] Hugo Riemann, *Musiklexicon*, 11th ed., s.v. "Euler."

[41] Carl Stumpf, "Konsonanz und Dissonanz," in *Beiträge zur Akustik und Musikwissenschaft*, vol. 1 (Leipzig: J. A. Barth, 1898), 22.

[42] Rudolf Schäfke, *Geschichte der Musikästhetik in Umrissen* (Berlin and Schöneberg: Max Hesse, 1934), 290.

was at odds with a burgeoning music psychology that rested on a better understanding of acoustics and a new emphasis on auditory physiology.

Euler was not without his supporters. Most, as Helmholtz and Oettingen, were scientists by profession, and several were involved with the early tone-psychological experiments. Although Stumpf had strong reservations about Euler's work, many of Euler's results agreed with his now famous *Tonsverschmelzung* investigations. Experimental work by Meinong and Witasek also seemed to confirm Euler's findings concerning octave gradation and the natural seventh.[43] Auerbach claimed that the particulars of Euler's system accorded with contemporary psychoacoustical research. Other natural scientists who stood by part or all of Euler's theory were Preyer, Meyer, Lipps, and Farnsworth.[44] The experimental results of these tone psychologists confirmed that natural sevenths were indeed "fuseable," and thus consonant as Euler had claimed.

Riemann's resistance to Euler had several facets. As a self-proclaimed *Harmoniker,* he traced his eighteenth-century roots to Rameau and believed in a natural rather than mathematical foundation of music. Rameau's theory had overshadowed Euler's in the eighteenth century, and its prestige among musicians grew in the nineteenth century. Riemann opposed a theory as out-of-tune with contemporary thought as Euler's, but did not limit his opposition to Euler. The tone psychologists also suffered under Rameau's mantle; but whereas Riemann at least knew

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43 A. Meinong and St. Witasek, "Zur experimentellen Bestimmung der Tonverschmelzungsgrade," *Zeitschrift für Psychologie und Physiologie der Sinnesorgane* 15 (1897): 196.

44 W. T. Preyer, *Über die Grenzen der Tonwahrnehmung* (Jena: H. Dufft, 1876); Max Meyer, "Zur Theorie der Differenztöne und der Gehörsempfindungen überhaupt," *Beiträge zur Akustik und Musikwissenschaft,* vol. 2 (Leipzig: J. A. Barth, 1898), 66–83; Theodor Lipps, *Psychologische Studien* (Leipzig, 1905); Paul R. Farnsworth, *The Social Psychology of Music* (Ames, Iowa: Iowa State Univ. Press, 1969).

the work of Stumpf, and was briefly optimistic about tone psychology, he seems to have been less familiar with or interested in Euler's work. Euler's name occurs once in connection with the Table (see 3.12), and scattered references—mostly disparaging—are found among the early writings, but there is nothing in Riemann that suggests a serious engagement with Euler's music theory. Perhaps he objected so strongly to the principles of this theory that he could not be bothered with its details. In any case, the generosity of spirit one finds in Helmholtz's appraisal of Euler is absent in Riemann.

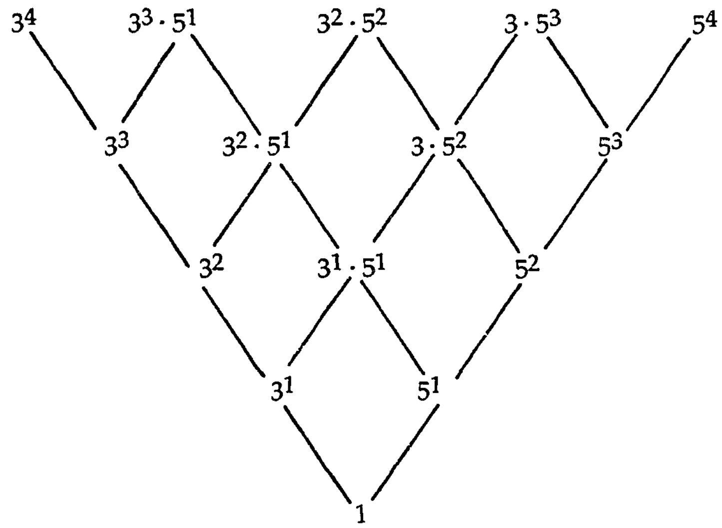
1.8 Formation of Tonal Genera

Euler's theory of scale formation relies on prime numbers in much the same way that his theory of consonance relied on prime forms of LCM. The theory of scale formation is also intimately related to the mirror of music. Indeed, Euler's scales or tonal genera reveal a unified theory of consonance, scale, and tonal relation.

Euler understood by the concept of musical genus a collection of pitches generated by a combination of prime factors and exponents—what we have been calling an *exponens*. This usage of *exponens* should not be confused with the mathematical term "exponent," since Euler's *exponens* indicate prime numbers as well as their powers. We briefly consider the *exponens* of three musically-related genera. The first is $2^n \cdot 3^3 \cdot 5^1$, which contains the pitch numbers: $3^0, 3^1, 3^2, 3^3, 3^0 \cdot 5^1, 3^1 \cdot 5^1, 3^2 \cdot 5^1$, and $3^3 \cdot 5^1$. Notice that Euler disregards powers of two since these only result in octave displacements. By translating the prime forms of $2^n \cdot 3^3 \cdot 5^1$ into frequency ratios within the octave 128 : 256, Euler obtains the series: 128 : 135 : 144 : 160 : 180 : 192 : 216 : 240 : 256. If F = 128 in this series, the resulting scale is F–F#–G–A–B–C–D–E–F. Euler calls this 8-tone scale with the *exponens* $2^n \cdot 3^3 \cdot 5^1$ *genus diatonicum.* He calls the 9-tone scale with the *exponens* $2^n \cdot 3^2 \cdot 5^2$ *genus chromaticum,* and the 8-tone scale with the *exponens* $2^n \cdot 3^1 \cdot 5^3$ *genus enharmonicum.* The three collections constitute genera 12, 13, and 14 of his system.

Hermann Busch, who has written the only extended study of Euler's music theory, models the eighteen genera of this theory by means of gridlike *Hassediagramme.* Like the Table of Relations, Busch's diagrams are open-ended and represent pure intervallic relations. The number 1 in Busch's model denotes a hypothetical reference tone (*Ausgangston*). Number 2 and its powers are omitted, since 2^n indicates octave relations. Busch's model, shown in Example 1-8, enables quick visual comparison of individual genera belonging to Euler's $2^n \cdot 3^a \cdot 5^b$ system of genera.

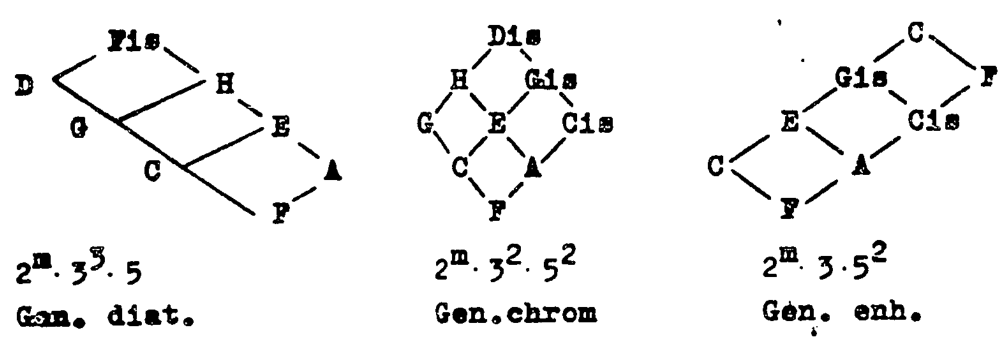
EXAMPLE 1-8: $2^n \cdot 3^a \cdot 5^b$ SYSTEM OF GENERA[45]



By means of this gridwork, one can easily locate pitches and their derivations for genera 12, 13, and 14. For example, if 1 = F, the remaining pitches of each genus correspond to specific products of 3 and 5 and their respective powers, in accordance with the *exponens* for that genus. For *genus diatonicum* ($2^n \cdot 3^3 \cdot 5^1$) therefore, 3^1 = C, 5^1 = A, 3^2 = G, $3^1 \cdot 5^1$ = E, and so on. This genus would occupy the portion of Example 1-8 consisting of all eight elements contained in $(2^n) \cdot 3^3 \cdot 5^1$. Genera 13 and 14 would occupy portions consistent with their *exponens.* Example 1-9 converts the prime numbers of Busch's model into letter names (1 = F), and shows the relation of the three genera to each other.

[45] Busch, 81.

EXAMPLE 1-9: RELATION OF GENERA 12, 13, AND 14[46]



In chapter 9 of *Tentamen,* Euler treats the 12-tone *genus diatonicum-chromaticum.* The content of this genus is exactly that of the *speculum musicum.* Busch abbreviates *genus diatonicum-chromaticum*—genus 18 in Euler's system—to GDC, a designation that we shall use throughout the remainder of this chapter.

GDC is produced from pure fifths and thirds by means of the following tuning process: F–C, F–A, C–G, C–E etc. or F–A, A–E, C–E, E–B and so on.[47] Euler first represented this process in a table, where he denoted fifth relations with the symbol "V" and third relations with the symbol "III". Example 1-10 reproduces this table as it appears in *Tentamen.* It is essentially the scheme that Euler used thirty-five years later in "De harmoniae veris principiis," where he simplified the format and added the title *speculum musicum.*

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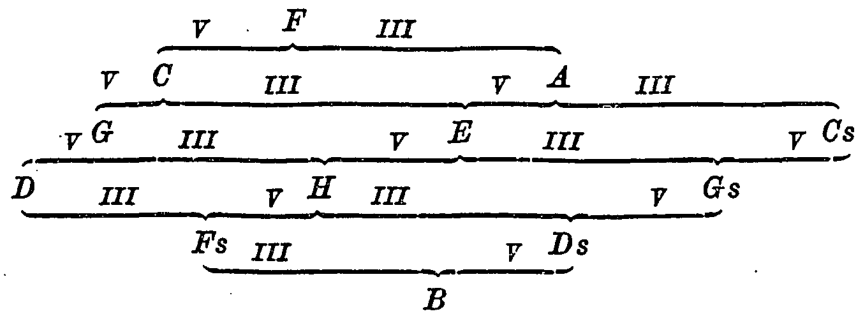
[46] Busch, 82.
[47] Busch, 82.

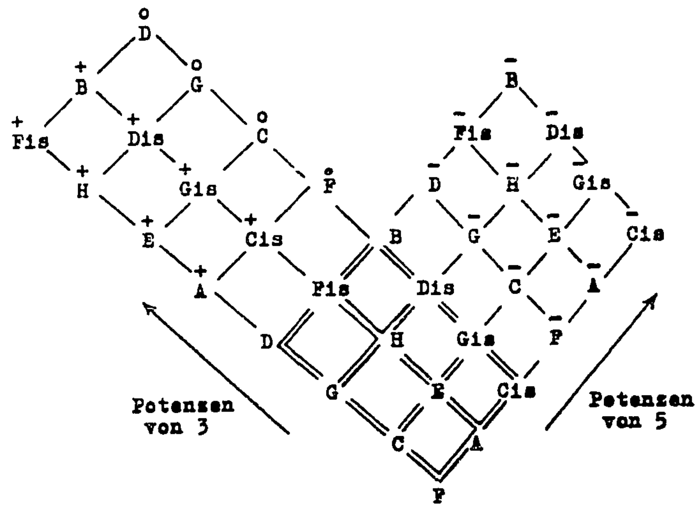
EXAMPLE 1-10: GDC TABLE AND *SPECULUM MUSICUM*[48]



Busch is quick to draw the connection between his diagrams and Euler's mirror. Genera 12, 13, and 14 each occupy different portions of the mirror, but by conflating 12 and 13, and adding the pitch B^b, all twelve tones of GDC are accounted for. Busch suggests that Euler's table may be extended ad infinitum by higher powers of 3 and 5. Euler himself never does this, but Busch argues that an extension is in keeping with the treatment of more complex genera in chapter 10 of *Tentamen.* Example 1-11 conflates genera 12, 13, and 14 and extends GDC (where 1 = F) by higher powers of 3 and 5.

[48] Euler, *Tentamen*, chap. 9, §13.

EXAMPLE 1-11: EXTENSION OF GDC[49]



The double-lined portion of this diagram is GDC or the mirror of music. Busch uses the symbol "+" to indicate pitches a comma (80 : 81) higher than their GDC counterparts. Similarly, "–" indicates pitches a diesis (125 : 128) lower than their unmarked counterparts, and "0" indicates pitches a diaschisma (2025 : 2048) lower.

Example 1-11 begins to look like Riemann's Table of Relations. Busch says that such geometric modeling represents the conceptual processes (*Gedankengänge*) at work in Euler's description of several complex genera.[50] The idea is that GDC is the simplest genus after the diatonic and chromatic genera, since it contains both of these genera and only one additional pitch. Through the combination of genera 13 and 14, or 12 and

[49] Busch, 83.
[50] Fuß (lix) notes anecdotally that Euler's "he dedicated his recreational hours to music; but even at the Clavier he heeded his geometrical imagination." [Erholungstunden widmete er der Tonkunst; aber auch an das Clavier beachte er seinen geometrischen Geist mit.]

14, however, new genera containing pitches outside GDC result, and these can only be represented by extending the *speculum musicum*. Euler considers these higher genera; the *genus chromatica-enharmonicum* ($2^n \cdot 3^2 \cdot 5^3$) contains three tones that do not belong to GDC (F, C, G), and the *genus diatonico-enharmonicum* ($2^n \cdot 3^3 \cdot 5^3$) contains four (F, C, G, D). He goes on to discuss still higher genera in *Tentamen,* including 24-tone divisions of the octave. In each case, the additional tones are simply the higher powers of 3 and 5 shown in Busch's diagram. Euler is careful to add that tones extraneous to GDC in his complex genera can be substituted with their GDC counterparts.

1.9 "The true principles of harmony" and Euler's "Mirror"

Euler remained committed to the relationship between tone and number throughout his career. The true principles of harmony were mathematical, but only in his final music treatise did Euler use the mirror to reflect these principles. One might expect the mirror to have been the focus of Euler's attention, but it was cited almost offhandedly near the end of the work. The reason is that the mirror represented nothing new at this stage, but instead synopsized a system that had been forty years in the making.

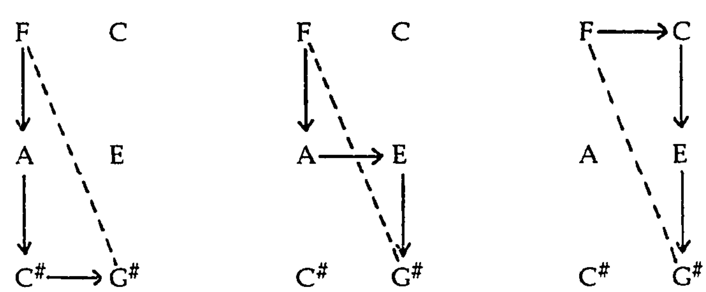
"De harmoniae veris principiis" is a brief work containing the intervallic theory belonging to GDC, and treating substitution and the natural seventh in passing.[51] Busch says the mirror is a formalism (*Schematismus*) that Euler introduced in this work to illustrate complex

[51] The treatise is 18 pages long in the modern edition: *Leonhardi Euleri Opera omnia,* ser. 3, no. 1:568–86.

intervallic steps as simply as possible. The mirror suggested how difficult intervals such as the tritone (F–B) might be conceived in terms of simpler intervals (F–C, C–G, G–B).[52] Busch also speculates that the mirror has "a certain suggestive value" for combinatorial applications in composition (*kombinatorische Spielvorgänge*), but does not elaborate what this value might be. Euler says little that pertains to either of these claims.

The principles underlying the formation of GDC are easy to illustrate with the mirror, and this seems to be the main use Euler intended for the model. Right-hand and left-hand movements (+V, –V) denote upper and lower fifths; upward and downward movements (+III, –III) denote upper and lower thirds. By conceiving intervallic relations vectorially—in terms of the line segments connecting the mirror's elements—complex intervals in GDC may be resolved into primes and represented graphically. Busch cites the interval 64 : 75, between F and G#, which may be represented three different ways on the mirror. Example 1-12 shows these possibilities.

EXAMPLE 1-12: THREE REPRESENTATIONS OF 64 : 75[53]



[52] Compare with Lewin's "Knight's move" where the tritone is understood against a subdominant–dominant progression; see Lewin, 21.
[53] Busch, 133.

Euler symbolizes the six major and minor triads of GDC in the same vectorial fashion His representation, seen in Example 1-13, is virtually the same as representations used a century later by Oettingen and Riemann to show the relation between major and minor harmony.[54]

EXAMPLE 1-13: REPRESENTATION OF MAJOR AND MINOR TRIADS[55]

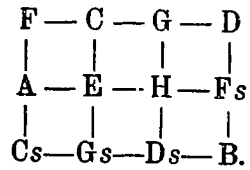
major minor

Euler concludes his discussion by posing two problems in relationship to the mirror. First, he asks how many trails lead from F to B♭ if the only permissible steps are +V and +III. He finds there are ten distinct trails and carefully lists each one.[56] We reproduce this list, along with the mirror, in Example 1-14.

[54] Compare with Hugo Riemann, *Systematische Modulationslehre als Grundlage der Musikalischen Formenlehre* (Hamburg: J. F. Richter, 1887), 173.
[55] Busch, 133.
[56] Euler, "De harmoniae veris principiis," 584, §29.

EXAMPLE 1-14: TEN TRAILS THROUGH THE MIRROR[57]

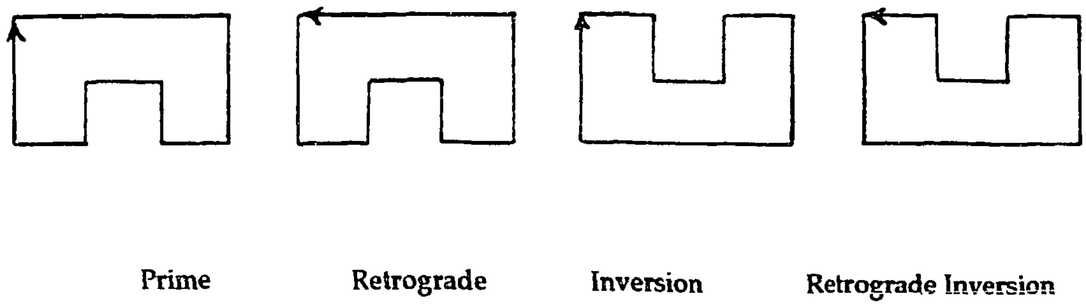
I. F : C : G : D : Fs : B,
II. F : C : G : H : Fs : B,
III. F : C : G : H : Ds : B,
IV. F : C : E : H : Fs : B,
V. F : C : E : H : Ds : B,
VI. F : C : E : Gs : Ds : B,
VII. F : A : E : H : Fs : B,
VIII. F : A : E : H : Ds : B,
IX. F : A : E : Gs : Ds : B,
X. F : A : Cs : Gs : Ds : B.



Euler then asks how many complete circuits from F to F are possible if the only permissible steps are ±V and ±III, and each pitch may occur just once. For this problem there are two solutions, although each circuit may be completed in either of two directions. Busch suggests a connection to serialism here, where each circuit is analogous to either the P, R, I, or RI forms of a 12-tone row. Example 1-15 sketches the four circuits. If the first is taken as a "prime" form, the second is its retrograde, and the third and fourth correspond to inversion and retrograde-inversion forms.

[57] Euler, "De harmoniae veris principiis," 584, §29.

EXAMPLE 1-15: EULER'S CIRCUITS THROUGH THE MIRROR[58]



Example 1-15 recalls a famous problem in topology that Euler had solved years earlier, and thereby reinforces the link between geometrical and tonal space in his music theory. The problem is the so-called "Seven Bridges of Königsberg." Königsberg was a Prussian city consisting of two islands, which were connected to each other and to the banks of the Pregel by a total of seven bridges. The approximate arrangement of land areas and bridges is sketched in Example 1-16a. Euler's problem was to determine whether there was any way to begin on one of the land areas (labeled A, B, C, and D), walk across each bridge exactly once, and return to the starting point. To solve the problem, he reduced the sketch in Example 16a to the graph in Example 1-16b. Land areas became vertices and bridges became edges in this graph. Euler was able to prove that a walk across each bridge and back to the starting point was impossible without backtracking; more generally, he proved that a complete circuit was impossible in any graph whose vertices did not have an even degree (were not the meeting point of an even number of edges). He also proved

[58] Euler, "De harmoniae veris principiis," 585, §30; Busch, 133.

that a course passing through every edge, but not returning to the starting point, was impossible in graphs with more than two vertices of an odd degree. These findings marked the beginning of a branch of mathematics that is known today as graph theory.

It is useful to distinguish the ways in which vertices may be traversed in Example 1-16b. The term "circuit" is used for a course that returns to its starting point, whereas "trail" is used for a course that does not return. A trail is thus an alternating sequence of vertices and edges in which each edge is flanked by its own vertices, no edge is used more than once, but at least one edge is used; 'd, db, b, e_3, a, e_2, c' is a trail on Example 1-16b. A circuit is simply a trail that begins and ends with the same vertex; 'a, e_3, b, e_4, a' is a circuit on Example 1-16b. A special kind of circuit that contains all the edges of a graph is called an "eulerian circuit"; the Königsberg problem amounts to showing whether the graph in Example 1-16b contains an eulerian circuit.[59] If we imagine the mirror of music as a graph consisting of twelve vertices and seventeen edges, we see that an eulerian circuit is impossible because six of the vertices have odd degrees (A, C, D#, F#, G, G#). The paths shown in Example 1-15 are both circuits, whereas those in Example 1-14 are trails.

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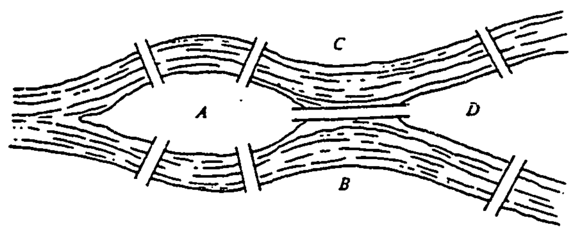
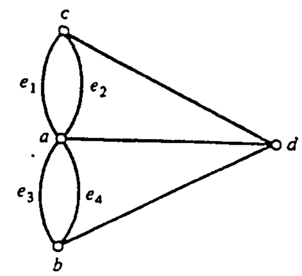
59 A circuit that does not repeat any vertex except the first and last is called a "cycle"; a "hamiltonian cycle"—after the Irish mathematician Sir William Rowan Hamilton (1805–65)—is a circuit that passes through all the vertices of a graph but one that might not use all the edges. The mirror contains a hamiltonian cycle but not an eulerian circuit.

EXAMPLE 1-16: KÖNIGSBERG BRIDGE PROBLEM

a) city of Königsberg



b) graph



It is also instructive to compare Euler's mirror of music with the musical circles that appeared in German treatises at the beginning of the eighteenth century. The famous circles of Mattheson and Heinichen were published in 1728 and 1735, respectively, and may have been known to Euler. Both the *speculum* and the early musical circles were representations of what their authors took to be the true principles of harmony. But whereas the mirror reflected an abstract system based on prime numbers, the circle-of-fifths summarized a concrete practice based on equal temperament. Busch's suggestion that Euler had actual music in mind when he posed combinatorial problems like those in Examples 1-14 and 1-15 seems fanciful. There is little evidence that Euler had any concern for the application of his theory, and he remained an opponent of equal temperament throughout his life. Unlike the musical circles, which remained tied to contemporary practice, Euler's mirror was a gloss on an impressive but ultimately speculative work. The association of the mirror with speculative theory, and the circle-of-fifths with applied theory, persisted in later variants of both. Riemann's Table of Relations also belongs with his speculative work, and stands in roughly the same relation to Weber's pitch space as the *speculum musicum* does to the circle-of-fifths. The speculative nature of the mirror and Table of Relations is an important line of continuity between Riemann and Euler.

1.10 Substitution

Substitution was mentioned earlier, in connection with the problems Euler encountered in his theory of consonance. The main idea was that human perception could "substitute" simple frequency ratios for complex ones in cases where the theory would otherwise assign too high a *GS*-rating. Euler did not explore the implications of this idea, but simply asserted it on an ad hoc basis to reduce inflated *GS*-ratings. Although his assertion of the idea was weak, the idea itself underlies an interesting philosophy of mind. Euler was essentially claiming that human perception had the agency to modify sense data, so that the structure of perceived objects (i.e. simple ratios) was potentially different from that of the objects themselves (i.e. complex ratios). One should not read too much Kant into this notion—Kant was younger than Euler after all, and published his important work late in life—but Euler's *Substitutionlehre* distinguished real-world objects from their representations in a manner not different in kind from Kant's distinction between phenomenal and noumenal modes of reality. We conclude this chapter with a brief account of Euler's notion of substitution.

Euler introduced substitution in "Conjecture," twenty-five years after the voluminous work of *Tentamen.* It is regrettable that he treated the concept as a corrective rather than as a basis for a theory of perception; as things stood, substitution could be applied without restriction to any acoustical event. In the case of the equal-tempered scale, which contained no natural consonances, one's ear presumably substituted whole-number ratios for the relationships between any two sounds in the system. Just intonation, in other words, served as a kind of perceptual default for equal temperament, which was widely in use by Euler's time. We shall see that Riemann entertained similar ideas about the relation between just and equal temperaments.

Substitution hinged mainly on Euler's preoccupation with the natural seventh, which he regarded as the characteristic feature of modern music. Writing about the apparent dissonance of seventh chords, Euler said that one "must carefully distinguish those ratios that our ears really perceive from those that the sounds expressed as numbers include." Chords with sevenths presented special problems for Euler. The LCM of their ratios tended to be large, which resulted in *GS*-ratings far out of proportion with the way these chords sounded. The notions of complete and incomplete consonance were of no help because the *GS*-rating could only be lowered by omitting the seventh altogether. Euler became particularly anxious to explain major-minor seventh chords. As we noted above (p. 20), the chord 36 : 45 : 54 : 64 received a *GS*-rating of 17, which was unaffected by the addition of 40 : 48 : 60. The rating seemed much too high but Euler worked around the problem by substituting the seventh 4 : 7 for 36 : 64. He believed that the relations contained in 36 : 45 : 54 : 64 were too difficult for most people to hear, and that simpler ones were probably perceived. Euler speculated that the ear substituted 63 for 64 so that 36 : 45 : 54 : 64 could be reduced to the much simpler 4 : 5 : 6 : 7, and 40 : 48 : 60 could no longer be inserted with impunity.[60] The difference between the actual chord and its substitute was small (slightly more than a comma), and the impression of the two chords was for all purposes identical. He called the substitute dominant a "belle harmonie" and attributed its special charm to the natural (substitute) seventh 4 : 7.[61] The chord was consonant in his mind, and required neither preparation nor resolution.

Through substitution Euler was able to solidify his theory of consonance. A sophisticated ear was needed to grasp harmonies based on the number 7, but Euler believed that his theory at least established the consonance of such harmonies. He believed that the mind had finally learned to count to 7 in music. The old objection that slight mistunings could turn the sweetest consonance into the harshest dissonance was also checked by the *Substitutionslehre.* As with seventh chords, Euler simply asserted that the mentally grasped relation was different from the absolute acoustical relation. This was the seed of a theory of perception that continued to blossom in the nineteenth century.

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[60] Euler, "Conjecture sur la raison de quelques dissonances généralement reçues dans la musique," 514, §14.
[61] Euler, "Du véritable caractère de la musique moderne," 533, §34.

CHAPTER 2: THE DUAL DEVELOPMENT OF HARMONY AND THE TABLE OF RELATIONS

2.1 Introduction

Riemann's mature harmonic theory rests on two ideas. One is dualism, or the idea that major and minor are opposite but equivalent systems. The other is function, or the idea that chords belong to general categories that determine chordal significance and progression within a key.[1] Riemann's three harmonic functions are tonic, subdominant, and dominant.

Dualism and function are independent ideas and emerged in Riemann's work as responses to different problems. The impulse to classify and generalize chords is evident as early as "Musikalische Logik" (1872) but Riemann did not settle on a technical language for this aspect of his theory until *Vereinfachte Harmonielehre* some twenty years later. Greater urgency surrounded the idea of dualism, for Riemann could not speculate freely on harmonic process until he had worked out a method for deriving chords. In this he was indebted principally to Hauptmann and Oettingen, both of whom he regarded as great dualists. The theory of harmonic function emerged comparatively slowly in his work, and at each stage of development had to accommodate an ever-present dualism.

Later we shall see how the marriage of function to dualism enhanced the Table of Relations. In this chapter we shall concentrate on dualism alone, specifically on the system introduced by Oettingen in his

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[1] Hugo Riemann, *Harmony Simplified; or The Theory of the Tonal Functions of Chords*, trans. H. Bewerunge (London: Augener, 1896), 9. Riemann writes: "there are three kinds of tonal functions (significance within the key), namely tonic, dominant, and subdominant." Also William C. Mickelsen, *Hugo Riemann's Theory of Harmony and History of Music Theory Book 3* (Lincoln: Univ. of Nebraska Press, 1977), 3–4.

Harmoniesystem in dualer Entwickelung (1866). Oettingen's work should be distinguished from the work of several theorists whom Riemann wishfully associated with dualism throughout his career. Riemann was wont to use history in this way, to prop up his own ideas, and in the case of dualism cobbled a path leading from Zarlino, Rameau and Tartini, to Hauptmann, Oettingen, and himself. The real dualists in this crowd were Oettingen and Riemann. Hauptmann's dialectical view of tonality inspired Oettingen, and greatly influenced Riemann, but his explanation of minor harmony—which rested upon a conceptual rather than perceptual distinction—was arguably not dualistic at all.[2] The pipeline for Riemann's dualism came directly from Oettingen. It was the language and method of his important but little known treatise *Harmoniesystem in dualer Entwickelung* that shaped Riemann's dualism most profoundly, and is key to understanding the Table of Relations in his early treatises. Oettingen's work reintroduced the Table—now as a model for an innovative brand of harmonic dualism—after it had been out of circulation for over a century.

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[2] Peter Rummenhöller writes: "there can be no question of dualism in Hauptmann's work, since all that matters—for major and minor chords, and for the entire system generally—is the principle of emergence through dialectical generation." [Von einer Dualität bei Hauptmann dann deshalb nicht die Rede sein, da für den Durakkord, wie überhaupt für das gesamte System, eben nur das eine Prinzip dialektisch-erzeugenden Hervorgehens gilt.]; see Peter Rummenhöller, "Moritz Hauptmann, Der Begründer einer transzendental-dialektischen Musiktheorie," in *Beiträge zur Musiktheorie des 19. Jahrhunderts*, ed. Martin Vogel (Regensburg: Gustav Bosse Verlag, 1966), 28. With this dialectical principle in mind, Daniel Harrison calls Hauptmann an "*existential dualist*...for whom oppositional structures were a pervasive and poetic fact of life"; see Daniel Harrison, *Harmonic Function in Chromatic Music: A Renewed Dualist Theory and an Account of Its Precedents* (Chicago and London: Univ. of Chicago Press, 1994), 243. The differences between Hauptmann's, Oettingen's, and Riemann's brands of dualism are addressed in 3.9 of the present study.

2.2 Arthur von Oettingen (1836–1920)

Oettingen is a peripheral figure in the history of music theory, and one whose work is all but forgotten today. He wrote a handful of works in which he elaborated harmonic dualism, and addressed some of the inconsistencies between his theory and musical practice. Oettingen was a consummate *Systemiker*, or system builder, and his treatises are models of order and presentation. In content, however, they are marred by a compulsion for symmetrical structure, which is played out to extreme lengths everywhere. Had dualism conformed better with musical practice, the place of honor assigned to Oettingen in "Ueber das musikalische Hören" might have been secured; but Riemann himself began pulling away from Oettingen soon after 1874, and his subsequent interest in Oettingen's work was minimal.

Interestingly, Oettingen was a physicist by profession and not a music theorist. His music-theoretic works reflect this, and if nothing else stand as monuments to the tremendous faith his culture placed in scientific method. The achievements of natural science in the late nineteenth century were numerous and so impressive that it was just a matter of time before someone—preferably a scientist—attempted to bring music theory into line with them. Oettingen was not the one to do this; the honor fell instead to his famous contemporary Hermann von Helmholtz, whose *Die Lehre von den Tonempfindungen* (1863) became an instant classic and a standard point of departure for Riemann's generation of music theorists. Helmholtz's researches into acoustics and auditory physiology opened up a new world for music theory, and theorists received this world with a mixture of gratitude and dismay; but however they felt about Helmholtz's specific results, his work clearly stirred them to action. With the possible exception of Hauptmann's *Die Natur der Harmonik und Metrik* (1853), no work exerted a stronger influence over German theory in the second half of the nineteenth century than *Die Lehre von den Tonempfindungen*.[3]

Oettingen's treatise of 1866 was written in the afterglow of Helmholtz's work, and may be construed as a 300-page correction of what many felt was Helmholtz's "error" concerning the minor triad. Minor harmony had been an annoyance to theorists for a long time, since it was foreign to the overtone series and derivable only through various theoretical sleights-of-hand. The weight of tradition was enough for most theorists to accept the minor triad, for even if they could not justify its use they were satisfied that no one could give reasons to justify its restriction. This state of affairs changed with Helmholtz, who embraced Euler's gradational notion of consonance, and set forth an acoustic theory wherein the minor triad was deemed more dissonant than the major triad (recall that major and minor triads received equal *GS*-ratings in Euler's theory). Helmholtz's findings were unpopular with musicians, and a little exasperating, since his data—the best that modern science and technology could offer—seemed unassailable. *Harmoniesystem* was the first serious rejoinder to *Die Lehre von den Tonempfindungen* and served to balance the dialogue for music theorists who lacked the scientific expertise to meet Helmholtz on his own terms. Harmonic dualism was essentially one physicist's critique of another's interpretation of complex and (for musicians) obscure scientific facts.

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[3] Leon Botstein, "Time and Memory: Concert Life, Science, and Music in Brahms's Vienna," in *Brahms and His World*, ed. Walter Frisch (Princeton: Princeton Univ. Press, 1990), 9–10.

Riemann learned of Oettingen's treatise shortly after his matriculation at the University of Leipzig in the Fall of 1871. An unfavorable review of the work in the *Neue Zeitschrift für Musik* caught his eye and rankled him enough to make him want to take a look for himself.[4] What interested Riemann was Oettingen's rationalization of minor harmony, which was refreshingly nonempirical; indeed, Oettingen seemed to be urging music theory to shake off its dependence on acoustical data and proceed according to its own logic. His treatise signaled a radical reform for Riemann and, along with Hauptmann's earlier work, inspired two predissertation essays that were published in installments in the *Neue Zeitschrift für Musik*.[5] To see how the Table supported Oettingen's ideas we must first back up to Hauptmann and clarify a more basic relation between dualism and musical space.

2.3 Dualism and Tonal Space

Harmonic dualism is inherently directional: Major triads are conceived as upward-generated structures, and minor triads as downward generated. The intervallic structure is identical—a major third and perfect

[4] See Wilibald Gurlitt, "Hugo Riemann (1849–1919)," *Akademie der Wissenschaften und der Literatur in Mainz: Abhandlungen der Geistes und Sozialwissenschaftlichen Klasse*, vol. 25 (1950): 1863–1905. Gurlitt (p. 1870) writes that Oettingen's treatise came to the younger theorist's attention "through a piece of malicious and disparaging criticism" [durch eine hämische absprechende Kritik]. For the review, see [J. S.?], "Ein neues Harmoniesystem," *Neue Zeitschrift für Musik* 65/42 (1869): 349–52.

[5] These are "Musikalische Logik: Ein Beitrag zur Theorie der Musik," *Neue Zeitschrift für Musik* 68/28–29, 36–38 (1872): 279–82, 287–88, 353–55, 363–64, 373–74; and "Ueber Tonalität" *Neue Zeitschrift für Musik* 68/45–46 (1872): 443–44, 453–54. Both essays were published under the pseudonym Hugibert Ries, and later reprinted in Riemann's *Präludien und Studien: Gesammelte Aufsätze zur Ästhetik, Theorie und Geschichte der Musik*, vol. 3 (Leipzig: Hermann Seeman, 1901). For a translation of "Ueber Tonalität," see Mark McCune, "Hugo Riemann's 'Ueber Tonalität': A Translation," *Theoria* 1 (1985): 132–50. Neither essay mentions dualism or the undertone series, but Riemann probably wanted to reserve these subjects for full disclosure in his dissertation.

fifth—the only difference being the direction in which the intervals are measured off. Oettingen was attracted by the mirror relation between major and minor triads and seized upon this as the basis for a comprehensive harmonic theory. Since his conception of major and minor was spatial, it was natural for him to seek a graphic format that depicted the symmetrical relation between them. The Table served this function well, as did other formats that Oettingen devised in connection with it.

Oettingen was not the first to conceive of major and minor triads spatially. Having rejected Helmholtz's account of these structures, he found welcome support for his ideas in the work of Hauptmann. Hauptmann's notion of harmony was very different from Oettingen's—his language exalted common chords to the grandeur of metaphysical forces and must have seemed particularly anachronistic to Oettingen—but like the dualists after him, Hauptmann too had a predilection for oppositional structures in music. (Unlike them, he did not insist that the fundamental of the minor triad was the fifth of the chord.) This predilection surfaced nowhere more clearly than in his explanation of the tonic and its two dominants. Dispensing with tedious ratios and appeals to nature to derive dominant harmony, Hauptmann invoked a simple opposition between a triad's *having* (*Haben*) a dominant and *being* (*Sein*) a dominant: C major has G major as a dominant, but is itself dominant to F major. Through such logic Hauptmann generated the three primary triads and the diatonic collection, which he laid out in the manner of Example 2-1.[6] The significance of Hauptmann's format, which

[6] Moritz Hauptmann, *Die Natur der Harmonik und der Metrik: Zur Theorie der Musik* (Leipzig: Breitkopf und Härtel, 1853), 27. In this example, and throughout our study, we shall use 'b' rather than the German 'h' to represent the pitch class B-natural.

emphasized the directional aspects of tonality, was not lost on Oettingen: tonic as center, subdominant left of center, dominant right of center. In less than fifteen years Oettingen would advance his own model as an improvement to Hauptmann's essentially one-dimensional or "flattened" version of the Table of Relations.

EXAMPLE 2-1: HAUPTMANN'S MODEL OF THE MAJOR KEY

I	III	II		I	III	II
F	a	C	e	G	b	D
		I	III	II		

Uppercase letters in Example 2-1 represent fifth relations (F–C, C–G, G–D); uppercase and lowercase together represent third relations (F–a, C–e, G–b). The raising of lowercase letters so that they form their own horizontal series ($F\ ^{a}\ C\ ^{e}\ G\ ^{b}\ D$) shows how small a step it is from Hauptmann's one-dimensional representation to the two-dimensional representation of the Table. The roman numerals in Example 2-1 denote the terms of Hauptmann's well-known dialectic: I = unity (*Einheit*), II = duality (*Zweiheit*), and III = union of duality (*Einheit der Zweiheit*, or *Verbindung*).

Having versus *being* permitted Hauptmann to explain minor harmony as well, and it was this explanation that sparked the fires of dualism and canonized Hauptmann for Oettingen, Riemann, and later dualists.[7] If a tone may be said to *have* a major third and perfect fifth, then

[7] See Chap. 3, n. 51 for an account of Oettingen's and Riemann's misreadings of the *Haben-Sein* polarity.

the opposite must also be true, namely, that a tone may *be* a major third and perfect fifth. The tone C *has* E and G as its third and fifth; by Hauptmann's logic, this same tone may be thought of as *being* the third of A^b and fifth of F. Major and minor triads are "opposites" in other words—one arises through having, the other through being had—though, again, Hauptmann did not claim that one actually hears the minor triad downward. The modeling of major and minor triads followed the format of Example 2-1, and is given below in Example 2-2.[8] The two representations of minor harmony reflect a conflict in Hauptmann's thinking. Example 2-2a is the diagram he normally used to show the "being had" relation; however, because he did not believe (as Oettingen and Riemann did) that the fifth of the minor triad was in any sense a root, he also gave the diagram in Example 2-2b, where C is interpreted as both *Zweiheit* of F and *Verbindung* of A^b—as a dialectical consequence of two pitches rather than an *Einheit* in itself. In each of Hauptmann's representations of major and minor harmony the *Haben-Sein* polarity found an explicit graphic image. These images became catalysts for new ideas of tonicity, phonicity, homonomy and antinomy that Oettingen would advance in the coming decade.

[8] Hauptmann, 35.

EXAMPLE 2-2: HAUPTMANN'S MODEL OF MAJOR AND MINOR TRIADS

a) minor triad			b) minor triad			c) major triad		
II		I				I		II
F	a^b	C	F	a^b	C	C	e	G
	III	I	I		II	I	III	
				I	III			

2.4 *Harmoniesystem in dualer Entwickelung*

We have said that Oettingen's language and method are key to understanding Riemann's early use of the Table. Let us now consider some of the novel terminology introduced in *Harmoniesystem*, beginning with the work's curious title. What exactly did Oettingen mean by the "dual" development of harmony? This was as obscure to nineteenth-century readers as for readers today, and Oettingen, sensing that "dual" and other unfamiliar terms might perplex music theorists, addressed terminology in a brief *Vorwort*.

At the risk of offending "ein philologisch gebildetes Ohr," Oettingen felt the need to introduce a terse language for ideas that would be awkwardly expressed otherwise. The word "dual" required special comment; though seldom used as an adjective in German, Oettingen claims it was "originally such a term" and uses it adjectivally in his title.[9]

9 The nominal form *Dualität* is more common. Goethe writes of the "the duality of appearance as antithesis" [Dualität der Erscheinung als Gegensatz] in his essay "Polarität," which was the basis for a lecture given in Weimar in 1805. The play of opposites is prominent throughout Goethe's work, and may have set a tone for mid-century German intellectuals. See Max Dufner and Valentine C. Hubbs, *German Essays II* (MacMillan: New York, 1964), 78–81.

Here dual refers not simply to the opposition between major and minor triads, but to further-reaching symmetries between the major and minor tonal systems. Opposition between the two triads was the germ of a more encompassing idea—a kind of logical primitive from which a fully developed theory might bloom. And though it was part of Oettingen's task to argue for the "innere Dualität" or oppositional equivalence of major and minor triads, his chief work was to elaborate the "äussere Dualität" or consequences of this idea.

The reference to logic is fitting, since "inner" and "outer" duality were intertwined propositions for Oettingen. He reasoned that the "inner duality or *Zweifältigkeit* of harmony also permits an outer, dual ... form of development for the harmonic system." This consisted in the "symmetrical construction of all tonal structures and harmonic progressions [and] it is in this sense that the word 'dual' is frequently used in the text."[10] Oettingen probably did not feel that inner duality was problematic, since most musicians already accepted the equivalence of major and minor harmony, and he devoted relatively little space to the subject in his treatise. Inner duality was the infrastructure of his theory and outer duality the superstructure; this outer dual development—the full-fledged and elaborate harmonic system to which the work's title refers—was Oettingen's preoccupation.

Why "dual"? Oettingen does not say why he chose this term when there were more conventional alternatives—"gegensätzlich" or

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[10] Arthur von Oettingen, *Harmoniesystem in dualer Entwickelung: Studien zur Theorie der Musik* (Dorpat und Leipzig: W. Gläser, 1866), iv. [Die innere Dualität oder Zweifältigkeit der Harmonie gestattet auch für das Harmoniesystem eine äussere, duale, d. h. zweifältig-gegensätzliche Form der Entwickelung, die in einem symmetrischen Bau aller Tongebilde und Klangfolgen sich kund thut. In diesem Sinne ist das Wort 'dual' auch im Texte häufig angewandt.]

"gegenseitig" for example—but it may help to reconsider the logical aspects of his conception. We shall see that Oettingen's system reflects at every turn the mirror relation between major and minor harmony: Outer duality is predicated strictly on inner duality, inner duality is reinforced in turn by outer duality. The ideas are interdependent and exert reciprocal force on each other.

There is another context in which the term dual denotes interdependence and reciprocity. This is in mathematical logic and involves the so-called "laws of duality," which assert that the complement of the union of two sets is equivalent to the intersection of their complements, and vice-versa. For any two sets K and L, the expressions $(K \cup L)' = K' \cap L'$ and $(K \cap L)' = K' \cup L'$ are said to be duals of each other.[11] The duality in these expressions results from the exchange of $\cup$ (union) for $\cap$ (intersection).[12] Consider the set $KL = \{f, a^b, c, e, g\}$, and the subsets $K = \{f, a^b, c\}$, and $L = \{c, e, g\}$. In musical terms, we may think of K and L as *Unter-* and *Ober-Klänge* of the composite *Klang* indicated by KL. Since the union of K and L is KL, the complement of this union relative to

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[11] We adopt the notation used in Alfred Tarski, *Introduction to Logic and to the Methodology of Deductive Sciences*, 2d ed. (New York: Oxford Univ. Press, 1946), 84. Tarski denotates the complement of a set K as K'. Readers may recognize in $(K \cup L)' = K' \cap L'$ and its dual $(K \cap L)' = K' \cup L'$ a generalization of De Morgan's laws, which assert for two propositions p and q, $\bar{}(p \wedge q) = \bar{}p \vee \bar{}q$, and $\bar{}(p \vee q) = \bar{}p \wedge \bar{}q$. W. V. Quine locates the essence of duality in these laws, named for the English logician Augustus De Morgan (1806–78), but credits a more deliberate treatment of duality to the German logician Ernst Schröder (1841–1902), whose three-volume *Vorlesungen über die Algebra der Logik* (Leipzig, 1895) systematically expanded the work of De Morgan; see W. V. Quine, *Methods of Logic*, 4th ed. (Cambridge, Mass.: Harvard Univ. Press, 1982), 84. Mathematical logic underwent a period of intense development beginning in the 1850s with George Boole's *An Investigation of the Laws of Thought* (London, 1854), and culminating in Bertrand Russell's and Alfred North Whitehead's *Principia Mathematica*, 3 vols. (Cambridge: Cambridge Univ. Press, 1910–13). Oettingen's *Harmoniesystem* appeared near the outset of this development and reflected, in its pursuit of a logical foundation for music theory, the orientation of much contemporary mathematical and philosophical thought.

[12] Quine (p. 81) writes that "the result of changing alternation to conjunction [the propositional equivalents of intersection and union] and vice-versa throughout [some schema] S is dual to S."

KL must be the null set, for which we would write $(K \cup L)' = \varnothing$. The complements of K and L relative to KL are $K' = \{e, g\}$ and $L' = \{f, a^b\}$, which means that the intersection of complements is also the null set: $K' \cap L' = \varnothing$, and therefore $(K \cup L)' = K' \cap L'$. In the dual expression $(K \cap L)' = K' \cup L'$, both sides of the equation come out to $\{f, a^b, e, g\}$.

Analogous exchanges obtain in Oettingen's formulation of inner and outer duality. Let us consider inner duality first. Oettingen explains the equivalence of major and minor triads by claiming that the tones of one are unified by a common fundamental, while those of the other are unified by a common overtone. He uses the term *Tonicität* to denote the property of intervals and chords sharing a common fundamental or tonic, and *Phonicität* to denote the opposite property of intervals and chords sharing a common overtone or phonic.[13] The major triad is tonically consonant but phonically dissonant, because its upper fifth does not occur among the initial overtones of its upper third (it eventually occurs as the nineteenth partial of the upper third). The reverse holds for the minor triad, which is phonically consonant but tonically dissonant because its lower third does not occur among the initial overtones of its lower fifth. (Remember that Oettingen forms minor chords downward from the fifth.)

[13] The definition of these terms in *Harmoniesystem* is as follows (pp. 31–32): "By the *Tonicität* of an interval or chord, I understand the property of being conceivable as a partial tone to some fundamental" [Unter Tonicität eines Intervalles oder Accordes verstehe ich die Eigenschaft desselben, als Klangbestandtheil eines Grundtones aufgefasst werden zu können]; "By the *Phonicität* of an interval or chord, I understand the property of possessing some partial tones that are common to all the tones of the interval or chord" [Unter Phonicität eines Intervalles oder Accordes verstehe ich die Eigenschaft desselben, stets irgend welche allen Tönen gemeinsame Partialtöne zu besitzen]. In the chord c-e^b-g, c and e^b *have* the common overtone g. This makes them undertones for Oettingen, but his sense of this term is different from Riemann's notion of an acoustical undertone. Oettingen writes that "one calls harmonic undertones all those tones that contain a given tone as overtone" (p. 31) [Harmonische Untertöne nennt man bekanntlich alle diejenigen Töne, die einen gegebenen Ton als Oberton enthalten]. The phonic explanation of minor harmony is an interesting distortion of Helmholtz, who also uses coincident partials to explain pitch relations but without any suggestion of dualism (see 4.3 of the present study).

Tonicität and *Phonicität* are dualistic concepts in the logical-mathematical sense stated above. The propositions behind these concepts are: (1) two partials share a common fundamental, and (2) two fundamentals share a common partial.[14] The difference between (1) and (2) is simply the exchange of "partial" and "fundamental." The propositions have the same symmetrical relationship as the symbolic expressions for the laws of duality. They also exploit Hauptmann's *Haben-Sein* polarity, although, unlike Hauptmann, Oettingen associates having with minor and being with major. This association is summarized in Example 2-3.

EXAMPLE 2-3: *HABEN* (*PHONICITÄT*) AND *SEIN* (*TONICITÄT*) IN OETTINGEN'S SYSTEM

Haben
(Phonality)

c–e♭–g have a common overtone of which they are fundamentals

Sein
(Tonality)

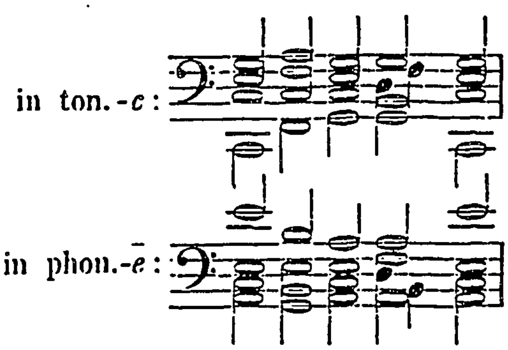
c–e–g are overtones of of a common fundamental

The dual development of harmony that ensued from Oettingen's notions of tonality and phonality was theoretically coherent but musically spurious. Oettingen took the major tonic system as his point of departure

[14] Oettingen worked extensively as an editor and translator, and was widely read in the scientific literature of the day. Harrison (pp. 246–47) suggests a connection between Oettingen's dualism and the work of French mathematician and engineer Victor Poncelet (1788–1867), whom he says "coined the term *dual* to describe the truth relationships that obtain when opposed terms are interchanged in some proposition."

and by adhering strictly to dualist principles, constructed a phonic system whose harmonic relations were not those of minor tonality or of any other system known to Western practice. Likewise, the dual of the diatonic major scale was not the diatonic minor but the Phrygian mode, whose high status in Oettingen's theory was not supported by compositional practice.[15] Example 2-4 encapsulates the relation between Oettingen's tonic and phonic systems. The upper portion shows a standard cadential progression in C major, or tonic 'c', and the lower portion shows its dual in phonic 'e' (which is not the same as A minor). Note the exchange of dominant and subdominant harmony in these progressions, as well as the mirror relation of voice leading.

EXAMPLE 2-4: THE MIRROR RELATION BETWEEN TONIC AND PHONIC SYSTEMS



Oettingen's correlate terms for subdominant and dominant harmony in the phonic system are *Unterregnante* and *Oberregnante*.[16] The

[15] Vestiges of the Phrygian mode survive in the frequent use of $\flat\hat{2}$ by later nineteenth-century composers. Oettingen (pp. 82–89) seems unaware of this and makes his appeal for the "phonische Geschlecht" on the basis of folk song, adding that "Nur aus der europäischen harmonischen Musik ist es verbannt worden" [it has been banished only from harmonic European music].

[16] Oettingen, 67. Oettingen needs these terms to round out the phonic side of his system. Like "dual", *Regnante* is also an unusual word in German. It is used somewhat archaically

Oberregnante e–g–b, with doubled fifth, is the second chord of Example 2-4. Two chords later comes the *Unterregnante* d–f–a in second inversion, which closes on the phonic a–c–e—a strange progression by traditional standards, but perfectly coherent in Oettingen's theory. Tonic and phonic triads built on the same pitch (f–a^{b}-c/c–e–g) are reciprocal (*Reciprocal*), a term that also describes the relation between mathematical expressions of duality. Reciprocal chords are formed oppositely and are therefore antinoms, whereas chords formed similarly are homonoms. We shall return to this terminology shortly, in connection with the Table, and in the next chapter where we shall see its importance for Riemann.[17]

2.5 The Table of Relations

Oettingen's treatise consists of an introduction and six chapters. The Table is presented early in the first chapter, and is invoked in later chapters when Oettingen discusses modulation and key relation. We shall focus on chapters 1, 3, and 4, which involve the Table directly and are most pertinent to Riemann's early work.

in English expressions, such as "the Queen regnant," to denote sovereignty. *Regnant* is the present participle of the Latin verb "regnare" ("to reign") and is an appropriate correlate for the word "dominant" in Oettingen's system.

[17] Riemann added the terms homolog and antilog to refer to the manner in which chords are connected. An intervallic ascent between two major chords—C major and E major for example—is *homologic* because the ascent follows the upward derivation of the chords themselves; an intervallic descent between minor chords is also homologic. An ascent between minor chords, or a descent between major chords, is *antilogic* because the direction does not follow the upward, or downward, orientation of the chords. With this quartet of terms—homonom, antinom, homolog, antilog—Riemann can classify any harmonic progression. A move from c–e–g to a^{b}–c–e^{b}, a case of simple modal mixture, is an *antilogic-homonomic* step: Both chords are major but the move to a^{b} is achieved through the descent of a major third.

Harmoniesystem begins with a synopsis of the history of theory since Rameau. Interestingly, Rameau is the only musician to whom Oettingen refers in this synopsis. The theorists who have made the greatest impression on him are either scientists or mathematicians. He cites the work of d'Alembert, Euler, and particularly Helmholtz. The reference to Euler is significant. Riemann, we recall, dismissed the number-based theory of Euler and other *Kanoniker* in favor of the perceptually oriented work of *Harmoniker*, among whom he counted Oettingen. Oettingen's reliance on number is striking, though, and his initial presentation of the Table as a means to show pure intervallic relations recalls Euler's use of the mirror in connection with the GDC model (see 1.8). Through Oettingen's influence a residual connection may be observed between Euler and Riemann, although Riemann doubtlessly would have resisted such a claim.

The magnitude of Helmholtz's influence, which we mentioned earlier in this chapter, comes through plainly in Oettingen's introduction. Oettingen credits Helmholtz with drawing into a coherent theory the scattered body of knowledge about partial tones and other physical attributes of sound. Among Helmholtz's other achievements, Oettingen lists his pioneering investigations into timbre, his study of human hearing, and his principles of *Klangverwandtschaft* (the relationship of chords) and *Klangvertretung* (the representation of chords by individual pitches). He shares Helmholtz's conviction that music theory must stand on a solid scientific footing before it begins to broach deeper psychological and aesthetic issues, and hails Helmholtz for "not merely [examining] the objective qualities of sounds, but also reducing the subjective activity of the human hearing organ, the mechanism of hearing, to rigorous physical concepts."[18] On the other hand, Helmholtz's ideas about partial tones and beating seem to Oettingen of greater value for orchestration than for a general theory of dissonance. Oettingen also believes that the dual relation between major and minor chords—unpursued by Helmholtz—bears on *Klangverwandtschaft* and *Klangvertretung* in a manner that invites full-scale revision of the tonal system, including chordal and key relations, and the theory of dissonance. Clearly his esteem for Helmholtz had limits, but it was Helmholtz's spirit—not Hauptmann's—that finally presided over his work. Oettingen remained true to natural-scientific values and was uncomfortable with the metaphysical aspects of Hauptmann's theory. Even where he did agree with Hauptmann, Oettingen believed that his own system was superior because it drew upon universal physical-physiological concepts rather than Hauptmann's more limited (*gering*) psychological ones.

Chapter 1 of *Harmoniesystem* introduces the *Buchstabentonschrift* (letter notation) that Oettingen uses in the Table and throughout the treatise. Oettingen explains how to derive the *Schwingungszahl* (frequency number) of specific pitches from this script, and vice-versa. There is also a consideration of intervallic consonance, and a brief treatment of the ratios of major and minor triads. Oettingen derives the "normal" or major tonal system in chapter 2, including its cadences and consonant harmonic structures. Chapter 3 treats chord progressions and

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[18] Oettingen, 2. [Helmholtz hat...nicht blos objectiv die Eigenschaften des Klanges, sondern auch die subjective Thätigkeit des menschlichen Hörorgans, den Mechanismus des Hörens, auf streng-physikalische Begriffe zurückgeführt.]

principles of modulation in closely related tonal systems. Chapter 4 presents the *Verwandtschaftskreis* (circle of relationship) for tonal systems of the same mode (homonomic systems) and for those of opposite mode (antinomic systems). Oettingen distinguishes two general kinds of relationship in this chapter, *parallele* and *reciproke*, and ascertains their boundaries. Chapter 5 treats tonal systems that result from modal mixture. Oettingen observes four "mixed" systems and compares these with the Greek modes. Chapter 6, the final chapter of Oettingen's treatise, is a theory of dissonance based on the dual principles elaborated in earlier chapters.

2.6 Tone and Number Reconsidered

The relation of tone to number preoccupies Oettingen in Chapter 1 of his treatise. His appeal to mathematics is nothing new for music theory, but unlike Euler and earlier *Kanoniker*, Oettingen's conception is harmonic and his starting point is the *Klang*, which he defines as "the total sensation of a periodic disturbance of air, consisting therefore of a series of tones."[19] Oettingen quotes several passages from Helmholtz, including the latter's claim that "sounds (*Klänge*) are distinguished from each other through their loudness (*Stärke*), absolute pitch (*Tonhöhe*), and timbre (*Klangfarbe*).[20] Absolute pitch is the only one of these attributes

[19] Oettingen, 21. [Die Gesammtempfindung einer periodischen Lufterschütterung heisst ein Klang. Es besteht also der Klang aus einer Reihe verschiedenartiger Töne.]

[20] Oettingen, 11. The term *Klang* presents special translation problems, since nineteenth-century theorists used it in an acoustical sense—i. e. "periodic disturbance of air"—as well as a music-theoretical sense. In references to specific chords, *Klang* normally appeared as part of a compound: *Unterklang, Oberklang, Terzklang, Quintklang*. We shall follow this practice as consistently as possible; where *Klang* appears on its own, however, the context will clarify whether one should read "chord" or "composite sound." See Chap. 5, n. 6 of the present study.

essential to dualism, and Oettingen dispenses with considerations of timbre and loudness as he develops his theory. *Klänge,* of course, may be expressed numerically: Since absolute pitch depends only on the frequency of Oettingen's "periodic disturbances of air," *Klänge* with higher frequency numbers will be higher in pitch than those with lower numbers. Like Euler, Oettingen uses whole numbers to express tonal relations in a system of pure or just intonation. His preference for just intonation is typical of the period and presumably, again like Euler, he considers it simpler and somehow more rational than equal temperament. The shared orientation toward number and pure tuning goes a long way in explaining the presence of the Table in Oettingen's treatise. Although Oettingen does not acknowledge Euler's mirror, it would be unusual for a scientist of his breadth not to have read Euler's work in acoustics and music theory. Oettingen's Table, however, was a perfect model in ways that would not have occurred to Euler. Its two dimensions were ideally suited to a chord-based theory of music, and Oettingen could represent the dual relationships of his system through the Table's symmetry. Example 2-5 reproduces the Table as it appears in chapter 1 of Oettingen's treatise. The term for representations such as this is *Darstellung,* a term to which we shall return in Chapter 4 of this study. Oettingen apologizes in his *Vorwort* (p. iii) for a work that is unfinished in some respects and presented merely in the form of a *Darstellung,* an ironic disclaimer given the importance of the Table to his theory.

EXAMPLE 2-5: OETTINGEN'S TABLE OF RELATIONS IN *HARMONIESYSTEM*

$5^m\ 3^n.$

n:	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8
m																	
2	$\overline{\overline{\text{c}}}$	$\overline{\overline{\text{g}}}$	$\overline{\overline{\text{d}}}$	$\overline{\overline{\text{a}}}$	$\overline{\overline{\text{e}}}$	$\overline{\overline{\text{h}}}$	$\overline{\overline{\text{fis}}}$	$\overline{\overline{\text{cis}}}$	$\overline{\overline{\text{gis}}}$	$\overline{\overline{\text{dis}}}$	$\overline{\overline{\text{ais}}}$	$\overline{\overline{\text{eis}}}$	$\overline{\overline{\text{his}}}$	$\overline{\overline{\text{fisis}}}$	$\overline{\overline{\text{cisis}}}$	$\overline{\overline{\text{gisis}}}$	$\overline{\overline{\text{disis}}}$
1	$\overline{\text{as}}$	$\overline{\text{es}}$	$\overline{\text{b}}$	$\overline{\text{f}}$	$\overline{\text{c}}$	$\overline{\text{g}}$	$\overline{\text{d}}$	$\overline{\text{a}}$	$\overline{\text{e}}$	$\overline{\text{h}}$	$\overline{\text{fis}}$	$\overline{\text{cis}}$	$\overline{\text{gis}}$	$\overline{\text{dis}}$	$\overline{\text{ais}}$	$\overline{\text{eis}}$	$\overline{\text{his}}$
0	fes	ces	ges	des	as	es	b	f	c	g	d	a	e	h	fis	cis	gis
−1	$\underline{\text{deses}}$	$\underline{\text{asas}}$	$\underline{\text{eses}}$	$\underline{\text{bb}}$	$\underline{\text{fes}}$	$\underline{\text{ces}}$	$\underline{\text{ges}}$	$\underline{\text{des}}$	$\underline{\text{as}}$	$\underline{\text{es}}$	$\underline{\text{b}}$	$\underline{\text{f}}$	$\underline{\text{c}}$	$\underline{\text{g}}$	$\underline{\text{d}}$	$\underline{\text{a}}$	$\underline{\text{e}}$
−2	$\underline{\underline{\text{bbb}}}$	$\underline{\underline{\text{feses}}}$	$\underline{\underline{\text{ceses}}}$	$\underline{\underline{\text{geses}}}$	$\underline{\underline{\text{deses}}}$	$\underline{\underline{\text{asas}}}$	$\underline{\underline{\text{eses}}}$	$\underline{\underline{\text{bb}}}$	$\underline{\underline{\text{fes}}}$	$\underline{\underline{\text{ces}}}$	$\underline{\underline{\text{ges}}}$	$\underline{\underline{\text{des}}}$	$\underline{\underline{\text{as}}}$	$\underline{\underline{\text{es}}}$	$\underline{\underline{\text{b}}}$	$\underline{\underline{\text{f}}}$	$\underline{\underline{\text{c}}}$

The outward appearance of the Table is quite different from that of Euler's mirror, but a reconsideration of its mathematical underpinnings will reveal some similarities. Oettingen was not happy to begin a treatise on music with a mathematical prolegomenon, and even invited readers to skip chapter 1 of *Harmoniesystem*.[21] Because this first chapter lays the groundwork for his theory and clarifies the Table's meaning, we may not pass over it so easily.

Oettingen first develops a method for assigning frequency numbers to the Table's *Buchstaben* script, and vice-versa. Instead of assigning a number that represents the actual frequency of central 'c', Oettingen lets c = 1 and determines the values for all other pitches in his system from this reference point. (This is essentially Euler's method, except F = 1 in his system.) The rows of fifths and columns of thirds may be extended infinitely without any repetition of pitch because the Table is based on whole number tuning ratios.

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21 David Lewin invites readers to do the same in his *Generalized Musical Intervals and Transformations* (New Haven: Yale Univ. Press, 1987). The Table is discussed briefly in chap. 2 of Lewin's work.

Oettingen begins by considering series of fifths. Since pure fifths may be expressed as powers of 3, and octaves as powers of 2, the frequency numbers for the central fifth-series are given by the formula $3^n \cdot 2^x$, where n and x are whole number exponents. We begin by fixing the value of x at 0, so that c = $2^0 = 1$. Different values of x will yield octave-steps above or below the central series, however, Oettingen cannot represent these in the two dimensions of the Table and does not bother to calculate them. If one were to imagine the central c ($2^0 = 1$) as middle C, or C4, then $2^1 = 2$ = C5, $2^2 = 4$ = C6, and so on. Because Oettingen does not explore such octave-series, we may understand 2^0 as the default for 2^x and concern ourselves only with the values of n in 3^n. For the letter names 'g', 'd', 'a', and 'e' these values are 1, 2, 3, and 4 (the n values for the Table's columns appear horizontally along the top of Example 2-5). The frequency number of 'g' is therefore $2^0 \cdot 3^1 = 3$; for 'd', 'a', and 'e' the numbers are $2^0 \cdot 3^2 = 9$, $2^0 \cdot 3^3 = 27$, and $2^0 \cdot 3^4 = 81$. Since every pitch is unique and no two letter names have the same frequency number, the idea of enharmonic equivalence is an immediate casualty of this system. Oettingen makes this point when he comments that "a power of 2 can never equal a power of 3."[22] Twelve fifth-steps to the right of 'c' will result in the pitch 'b#' ($2^0 \cdot 3^{12} = 531441$). Because 'b#' is a power of 3 (that is, part of a fifth-series), and higher octaves of 'c' are powers of 2, 'b#' ≠ 'c'. The acoustical difference between this 'b#' and the nearest 'c' ($2^{19} \cdot 3^0 = 524288$) is small—only slightly greater than a syntonic comma (the ratio 524288 : 531441 is equal to 23.5 cents)—but the conceptual difference is very great: 'c' occupies a hypothetical third dimension nineteen octave-steps above the Table's central 'c'; 'b#' is a distant point twelve fifth-steps right of center. The derivation of any 'c'

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[22] Oettingen, 12.

will always be different from that of any 'b#'. Indeed, every pitch will have a unique relation to central 'c'. One could use coordinate pairs as pitch addresses in this system—'c' = (0, 0), e = (0, 1), g = (1, 0) and so on. (These addresses would naturally exclude octave-series, but the fact that Oettingen models his theory in just two dimensions suggests that pitches related by octave are in some sense equivalent for him.)[23] Pitch addresses could be used to measure distance on the Table, but it is doubtful that such measurement would shed light on intuitions of perceived distance in music. There are countless paths between any two locations on the Table, and it is not at all clear what advantage might be gained by affixing distances between these locations.

Third relations are less important for Oettingen than fifth relations. This is reflected by a predominantly horizontal conception of the Table. The Table's columns serve essentially as conduits leading to and opening up more remote series of fifths—upper and lower thirds are never ends in themselves. When Oettingen considers third-related series of fifths, however, he encounters letter-name redundancies, and this prompts a discussion of third-relations and a modification of his script. Coincidences of letter name are unavoidable but do not represent actual acoustical redundancy. Oettingen explains them as the results of differences between Pythagorean intonation—which is based on pure fifths—and just intonation, which is based on pure fifths and pure thirds. For example, 'e' may be thought of as the upper third of 'c' or as the fourth upper-fifth. The first 'e' is a power of 5 ($5^1 \cdot 3^0 = 5$), whereas the second is a power of 3 ($5^0 \cdot 3^4 = 81$). Although the letter names are the same, the two intervals 'c–

[23] Lewin calls the Table a "modular harmonic space" (p. 21) because it strips away octave relations and, in effect, folds three dimensions into two.

e′ will sound different. The first is a pure or just third and the second is a Pythagorean third.

To indicate pitch deviations between letter-name pairs, Oettingen incorporates Helmholtz's *Horizontalstriche*.[24] A horizontal line above a pitch's letter name means the pitch is one syntonic comma lower than its unlined counterpart. The just third 'c–ē' is therefore smaller than the Pythagorean third 'c–e' (the difference between the intervals is about 21.5 cents). A horizontal line below a pitch's letter name means the opposite. The further one moves from the central series, the more *Striche* one needs to represent these tuning discrepancies. Besides tuning discrepancy, however, the *Striche* also show position: Two upper *Striche* denote a fifth-series two vertical steps "north" of 'c'; two lower *Striche* denote a series two steps "south" of 'c'. By means of *Striche*, one gains an immediate sense of position relative to the Table's central series.

Once Oettingen has explained the tuning relations of the Table, he is ready to generalize the relation between tone and number. Because thirds, fifths, and octaves may be expressed as powers of 5, 3, and 2, respectively, the formula $5^m \cdot 3^n \cdot 2^x$ will determine the *Schwingungszahl* for any tone in the system. Oettingen abbreviates this formula to $5^m \cdot 3^n$, since powers of 2 are excluded from the Table (he proposes using vertical *Striche* and uppercase letters to distinguish octave-related pitches, but incorporates neither into the Table). For any tone t, m may be thought of as the number of third-steps north or south of the central series, and n the number of fifth-steps east or west of the origin. The Table's m values in Example 2-5 are limited to horizontals two third-steps above and below

[24] Helmholtz introduced *Striche* in the first edition (1863) of *Tonempfindungen* as a modification to Hauptmann's notation. See Chap. 3, n. 62 of the present study.

the central series: $m = \{-2, -1, 0, 1, 2\}$. The n values are slightly more inclusive: $n = \{-8, -7 \ldots 7, 8\}$. To determine the *Schwingungszahl* for '$\bar{c}$'—one step north, four steps west of the origin—one would fix m at 1 and n at -4, and apply the formula: $5^1 \cdot 3^{-4} = {}^5/_{81}$. For the dual pitch '$\underline{c}$'—one step south, four steps east of the origin—one would take the opposite values of m and n, and apply the formula: $5^{-1} \cdot 3^4 = {}^{81}/_5$. In general, where m is positive and n is negative in $5^m \cdot 3^n$, the *Schwingungszahl* $= \frac{5^m}{3^n}$; where m is negative and n is positive, the *Schwingungszahl* $= \frac{3^n}{5^m}$; where both exponents are positive, the *Schwingungszahl* $= \frac{5^m \cdot 3^n}{1}$, and where both are negative, the *Schwingungszahl* $= \frac{1}{5^m \cdot 3^n}$. Notice the numerical relation between pitches sharing the same letter name. We have seen that '$\bar{c}$' = ${}^5/_{81}$; by applying the formula to '$\bar{\bar{c}}$' ($m = 2, n = -8$), we get ${}^{25}/_{6561}$, and by applying it to '$\bar{\bar{\bar{c}}}$' ($m = 3, n = -12$), we get ${}^{125}/_{531440}$. In general, for any $l^m = {}^a/_b$ (where l = letter name, $m \neq 0$, and a and b are positive integers), $l^{m+1} = \frac{5a}{81b}$, and $l^{m-1} = \frac{81a}{5b}$. The prominence of the number 81 in this relationship is due to the syntonic comma, whose ratio 80 : 81 represents an intervallic constant between like-named pitches. Oettingen remarks that no single pitch can be expressed by more than one combination of integers standing for m, n, and x in the expression $5^m \cdot 3^n \cdot 2^x$, since every number has a unique prime form. Ultimately, *Schwingungszahlen* and *Buchstaben* are correlated with prime forms, which we recall played an important role in Euler's theory too. Later we shall see Riemann musing over prime numbers in connection with the Table.

The upper-left to lower-right diagonals of the Table form series of minor thirds. The minor third (which Oettingen calls the *Ergänzung*), complements the major third, and is the difference between the interval classes represented in the Table's horizontal and vertical series. Oettingen adds these two interval classes to form the major seventh, and finds seventh-series in the opposite diagonal, which contains all of the ascending and descending leading tones of a key. By means of the minor-third diagonals, Oettingen is able to use the Table to represent "alle reinen consonanten Dreiklänge," in the form of right-angle triangles. These triangles stand in a mirror relation to each other—the major triangle points upward, and the minor triangle points downward—and perfectly depict Oettingen's conception of inner duality.

Oettingen compares his notation to that of Helmholtz and Hauptmann, who both use uppercase letters for fifth-related pitches (C–G, G–D, D–A) and an uppercase-lowercase combination for third-related pitches (F–a, C–e, G–b). Helmholtz and Hauptmann thus represent the supertonic d minor as D–F–a, to show that D and F are part of one series and 'a' is part of another. Oettingen is able to dispense with uppercase letters because the Table makes the derivation of the chord clear: 'd' and 'f' lie along the central fifth-series; 'ā' is a pure third one step north of 'f'. Oettingen thus writes d–f–ā instead of D–F–a.[25] Since the interval d–a is 21.5 cents smaller than the pure fifth d–a, and since 'd' is not adjacent to f–a on the Table, the chord d–f–a is a dissonance in his system. Oettingen depends on just intonation in this way to make distinctions between chords that are nominally the same but acoustically different. He appeals to Euler's work, as well as Hauptmann's and Helmholtz's, and seems dismayed that so much music theory has ignored the subject of

[25] Helmholtz saw the advantage of this method, and did away with uppercase *Buchstaben* in the third edition (1870) of *Tonempfindungen*. While Oettingen used upper *Striche* to denote the lowering of pitch, and lower *Striche* to denote the raising of pitch, Helmholtz (and Riemann after him) did just the opposite; see Chap. 3, n. 62.

intonation. There is no turning away from pure tuning Oettingen believes, once "the essential nature of Helmholtz's researches" has been penetrated.

2.7 Tone-System Relations

Oettingen generalizes the relations among tonic and phonic *Tonsysteme* in the third chapter of *Harmoniesystem.* Essentially he is treating key relations in this chapter, but the word "key" is better avoided given the nature of the phonic system. The occurrence of parallel chords between systems—chords derived from the Table of Relations in the same manner—is an important determinant of tone-system relations. So too is the occurrence of reciprocal chords (opposites such as c–e–g and f–$\underline{a}^{b}$–c). Oettingen thus invokes the notions of homonom and antinom (see p. 56 above) to help explain tonic and phonic relations. Chords that are formed similarly are homonomic, whereas chords formed oppositely are antinomic. Parallel chords are homonomic, in other words, and reciprocal chords are antinomic.

Oettingen extends the meaning of homonom and antinom to tone-systems; the relation between two major (tonic) systems is homonomic, whereas that between a major and a minor (phonic) system is antinomic. Every tone-system has three homonomic and two antinomic chords; the nearest degree of relation obtains when the tonic or phonic chord of one system is represented identically in another. Oettingen uses the term *Klangvertretung* to refer to such representations of parallel chords. (Riemann uses the same term synecdochically, to refer to single pitches that stand for entire chords.) For any system, there are two homonomic and two antinomic systems in which the tonic or phonic triad will appear. Tonic 'c' is related through this notion of *Klangvertretung* to tonic 'f' (*Unterdominante*), tonic 'g' (*Oberdominante*), phonic 'ē' (*Parallel*), and phonic 'b̄' (*Leit-Tonart*) because the tonic triad c–ē–g occurs in each of those four systems. To decide which of them is closest to tonic 'c', Oettingen first considers tuning discrepancies but then decides that common chords are a much better measure of relatedness. Mere representation is not sufficient, though. Oettingen also considers the meanings of common chords in each related system; the number and meaning of such chords determine a system's relatedness to tonic 'c'.

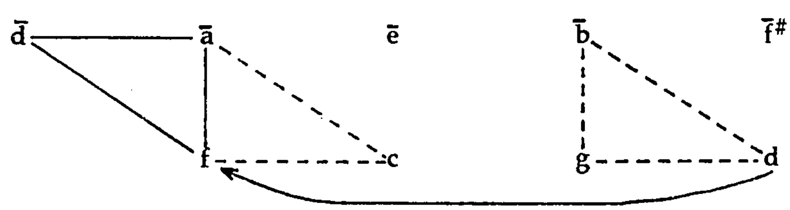
Chordal meaning (*Bedeutung*) is an undernourished concept here, since Oettingen does not classify chords functionally. (By contrast, *Funktion* and *Bedeutung* are more or less interchangeable for Riemann.) Chords are either essential *Hauptklänge* or nonessential *Nebenklänge*. In tonic 'c' the *Hauptklänge* are *Tonica* c–ē–g, *Unterdominante* f–ā–c, and *Oberdominante* g–b̄–d; the *Nebenklänge* are *Oberterz* ā–c–ē, and *Leitklang* ē–g–b̄. In phonic 'ē', the *Hauptklänge* are *Phonica* ā–c–ē, *Unterregnante* d̄–f–ā, and *Oberregnante* ē–g–b̄; the *Nebenklänge* are *Unterterz* c–ē–g, and *Leitklang* f–ā–c. Although Oettingen distinguishes among essential and nonessential chords, his theory does not assign them specific contextual roles. Terms such as *Unterterz* and *Leitklang*—and even *Tonica* and *Oberdominante*—are labels that describe chord derivation, not function. A chord's meaning is limited therefore to rather trivial categories of "essential" and "nonessential." Such nonessential chords as Oettingen's tonic and phonic *Leitklänge* had to await Riemann for their functional clarification. (Both are tonic *Leittonwechselklänge* in Riemann's system; see 5.4 and 5.5). Indeed, all of Oettingen's *Hauptklänge* and *Nebenklänge* were assimilated into Riemann's function theory, in addition to much of the terminology he used to describe them. Oettingen's theory is in this sense proto-Riemannian, and lacks only the functional component. Example 2-6 summarizes the relationship between tonic 'c' and its four closest systems. An analogous table could be drawn up for phonic 'ē', whose closest relations are tonic 'c', tonic 'f', phonic 'ā', and phonic 'b̄'.

EXAMPLE 2-6: TONE-SYSTEMS CLOSELY RELATED TO TONIC 'C'

Tone-System	Common Chords	Meaning in Tonic 'c'	Meaning in Tone-System
1) phonic 'ē'	ā–c–ē	Parallel	Phonica
	ē–g–b̄	Leitklang	Oberregnante
	c–ē–g	Tonica	Unterterz
	f–ā–c	Unterdominante	Leitklang
2) phonic 'b̄'	ē–g–b̄	Leitklang	Phonica
	ā–c–ē	Parallel	Unterregnante
	g–b̄–d	Oberdominante	Unterterz
	c–ē–g	Tonica	Leitklang
3) tonic 'f'	f–ā–c	Unterdominante	Tonica
	c–ē–g	Tonica	Dominante
	ā–c–ē	Parallel	Leitklang
4) tonic 'g'	g–b̄–d	Oberdominante	Tonica
	c–ē–g	Tonica	Unterdominante
	ē–g–b̄	Leitklang	Oberterz

Example 2-6 omits several chords from consideration. Phonic 'ē' and tonic 'c' both contain a d-minor chord, for example, but d minor is not listed as a common chord between these systems. The reason for this omission and others concerns tuning. In just intonation, the d-minor chord in question is actually two different chords. The *Unterregnante* of phonic 'ē' contains the pitches d̄–f–ā, which fall into triangular formation on the Table, whereas the d–f–ā chord of tonic 'c' does not fall into triangular formation. Example 2-7 shows the derivation of these two chords; the second is actually an amalgam of the chords f–ā–c and g–b̄–d. The 'd' belongs to the central series as the fifth of 'g', and sounds slightly out-of-tune with the nonadjacent interval f–ā. This d–f–ā chord is Rameau's chord-of-the-added-sixth, a subdominant f–ā–c with 'd' substituting for the omitted fifth. We earlier represented it as D–F–a; using this notation, one would write D–f–A for the *Unterregnante* d̄–f–ā in phonic 'ē'. The *Unterregnante* is perfectly consonant, whereas the chord-of-the-added-sixth is an example of what Riemann calls a *Scheinkonsonanz* or "apparent consonance." Oettingen characterizes the modulation from tonic 'c' into phonic 'ē' chiefly as a move from the dissonant d–f–ā to the pure phonic d̄–f–ā.[26]

EXAMPLE 2-7: CHORD-OF-THE-ADDED-SIXTH (d–f–ā) AND *UNTERREGNANTE* (d̄–f–ā)



26 Hauptmann went so far as to call the dissonant d–f–ā chord a diminished triad. Oettingen does not claim this, but feels nonetheless that his predecessor's insight into the double nature of the supertonic triad has not been fully appreciated. For example, Heinrich Bellerman takes up the subject in connection with Sechter's theory but makes no mention of Hauptmann, whose theory, according to Oettingen "scheint dem Verfasser [Bellerman] nicht bekannt zu sein" (p. 81); see also Heinrich Bellerman, *Der Contrapunkt; oder, Anleitung zur Stimmführung in der musikalischen Composition* (Berlin: Julius Springer, 1862), 8.

Although phonic 'ē' and 'b̄' each share four chords with tonic 'c', they are not as closely related to this system as the tonic systems of 'f' and 'g'. The relation between tonic 'c' and phonic 'ē' (the *Paralleltonart* or relative minor) is weakened because the *Oberdominante* g–b̄–d does not occur in phonic 'ē', and the *Regnante* d̄–f–ā does not occur in tonic 'c'. Besides the tonic and phonic chords, these are the most important *Hauptklänge* of the two systems. Their omission in Example 2-6 is not trivial and may explain why Oettingen judges phonic 'b̄', the *Leittonart,* as only slightly less related than phonic 'ē' to 'c'. The *Leitton* relation is a novel aspect of Oettingen's theory, and one he believes traditional theory has obscured by insisting on a third-relation between chords such as c–ē–g and ē–g–b̄. The Table clarifies the true relation between these chords, and shows how close they are indeed: One diagonal "leading-tone step" separates c–ē–g from ē–g–b̄. The leading-tone 'b̄' is in effect substituted for the root 'c' to yield the chord ē–g–b̄. Oettingen follows Hauptmann with respect to the *Unterdominante* and *Oberdominante* systems in Example 2-6: Of the two tonic systems, he believes 'f' is closer than 'g' to tonic 'c' because 'c' has a gravitational pull toward 'f' as its upper dominant.

2.8 Chord Connection

An unusual and remarkable theory of chord progression underlies Oettingen's treatment of modulation. Chord progression is "transformational" for Oettingen—the *Leitklang* ē–g–b̄ is the result of something one does to c–ē–g, in this case the result of applying the antinomic *Leittonschritt* to the tonic. Oettingen sometimes uses the word *Umwandlung* to describe chord progressions, which stresses the relation between chords but does not connote the progressive aspect of harmony.[27] The term "connection" probably describes Oettingen's transformational approach to chords better than "progression," and we shall use it throughout our discussion. The model of chord connection is of great importance because it shows us how Oettingen navigates the Table of Relations. The Table assumes new significance once Oettingen begins addressing chords and modulation. Besides showing tuning relations among pitches, it now shows transformational possibilities among chords.[28] In the course of discussing chord connections, Oettingen considers more distantly related tone-systems than the ones appearing in Example 2-6.

Oettingen claims that third- and fifth-relations between chords are directly intelligible, but unlike Helmholtz (see 4.3) he dispenses with common tones as a measure of chord relation: The closest chords to tonic 'c' are its upper and lower dominants (g–b̄–d; f–ā–c), followed by the phonic chords of the upper third (ā–c–ē) and leading tone (ē–g–b̄). Any of these chords may follow the tonic directly with good effect. Indeed, they are all the results of transforming the tonic in the sense suggested above. The homonomic *Quintschritt* yields the chords g–b̄–d and f–ā–c, the antinomic *Terzschritt* yields the chord ā–c–ē, and the antinomic

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27 The term *Fortschritt*—a "stepping forth"—is normally used by German theorists, but Oettingen prefers to speak of *Klangfolgen*, or harmonic successions. The third section of *Harmoniesystem* is entitled "Modulation und Klangfolge."

28 Riemann's theory is also transformational in this sense. See Chap. 5 of the present study.

Leittonschritt—a compound fifth-plus-third step—yields the chord $\bar{e}$–g–$\bar{b}$.[29] To these transformations Oettingen adds one of particular importance, which he calls the antinomic *Wechsel* or *Octavschritt*. The *Wechsel* transforms a chord into its reciprocal (c–$\bar{e}$-g into f–$\underline{a}^{b}$–c), which may also follow the tonic directly. Without the antinomic *Wechsel* most chords in the Table would remain inaccessible (Oettingen mentions $\bar{b}$–$\bar{\bar{d}}^{\#}$–$\bar{f}^{\#}$, the reciprocal of $\bar{e}$–g–$\bar{b}$), but all are within reach once it is incorporated. The antinomic *Wechsel* is an important mediating step between indirect (nonadjacent) chords, and thus for modulation into remote tone-systems. One has only to apply it once to move into a new fifth-series, and through successive combinations of antinomic *Terzschritte* and *Wechsel* any fifth-series in the Table may be reached. Oettingen claims that vertical moves to homonomic chords a major third apart are frequent, and much nineteenth-century music bears out his claim. The strength of the transformational system is that it can mediate the remotest of connections with a repertoire of just four moves. Example 2-8a summarizes these moves, and Example 2-8b demonstrates how one may proceed from one fifth-series to another through applications of T (antinomic *Terzschritt*) and W (*Wechsel* or *Octavschritt*).

[29] Notice that Oettingen does not distinguish the direction of the *Quintschritt*. For Riemann, the move to *Oberdominante* was a homologic-homonomic *Quintschritt*, whereas the move to *Unterdominante* was an antilogic-homonomic *Quintschritt* (see n. 17)

EXAMPLE 2-8: BASIC TRANSFORMATIONS; MOVING AROUND WITH T AND W

a)

Basic Transformations

1) homonomic Quintschritt (Q): c+ – g+; $\overline{e}$° – $\overline{a}$°

2) antinomic Terzschritt (T): c+ – $\overline{e}$°

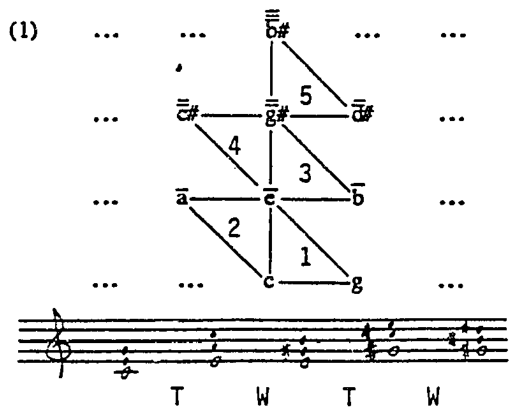
3) Wechsel (Octavschritt) (W): c+ – c°

4) antinomic Leittonschritt (L): c+ – $\overline{b}$° [Q + T = L]

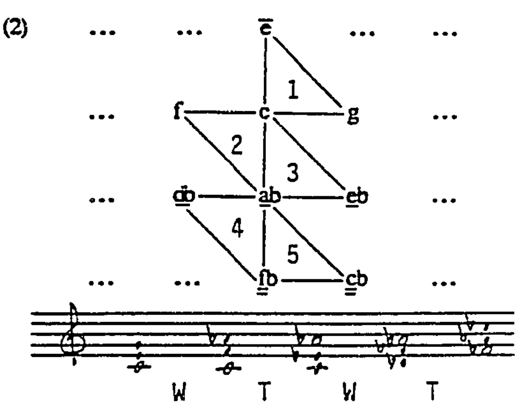
b)

Moving Around With T and W

(1)



(2)



An important feature of Example 2-8b is that it shows how Oettingen conceives of chromatic third relations between chords such as C major and E major, or F^b major and A^b major. A chromatic third relation, or instance of secondary mixture, is indirect and the chords involved must be mediated through some combination of transformations.[30] In the move

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30 Secondary mixture is Aldwell and Schachter's term for "the alteration of the 3rd of a triad where such alteration does not result from normal mixture." See Edward Aldwell and Carl Schachter, *Harmony and Voice Leading*, 2d ed. (New York: HBJ, 1989), 363–64.

from C major to E major an A minor chord provides the necessary connection, for it is directly related to C major through T, and to E major through W.

Through different combinations of Q, T, W, and L there is no limit on the number of paths one can trace between two chords. The model of chord connection is so versatile that it can account for any chord progression in common practice tonality. Often, however, the account is unsatisfying. Two particular weaknesses (which Riemann later remedies) frustrate attempts to find musically coherent paths on the Table. One is Oettingen's reliance on just intonation, which he did not consider a weakness at all, and the other is his failure to assign functions to chords, which he did consider a weakness but could not circumvent.

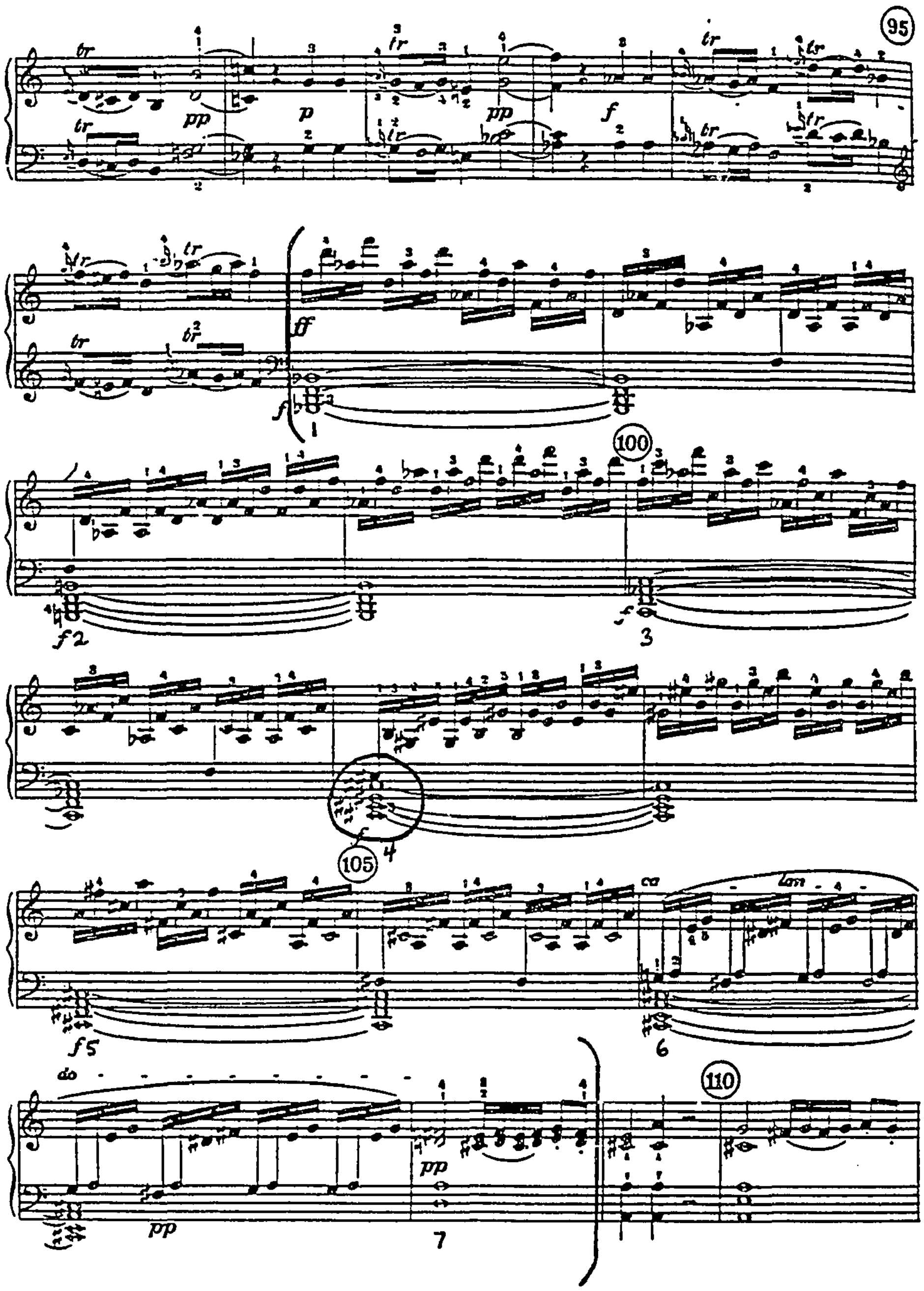
We have mentioned both of these problems, but have yet to see their consequences in a musical situation. Because of just intonation, Oettingen's model is insensitive to chordal ambiguity brought on through enharmonic change. Chords, like pitches, have unique acoustic identities and derivational positions on the Table. No chord can "be two places at once," yet composers routinely exploit just such ambiguity. The problem of function is intertwined with Oettingen's reliance on just intonation, for it would be daunting indeed to develop a functional language for a system containing an infinitely large collection of chords. Oettingen is not up to the task, and leaves it to future generations "to sift through the chaos of possibilities and create order by means of a clearly recognized law through which the higher freedom of art is achieved."[31] Though Oettingen

31 Oettingen, 156. [Es wird eine Hauptaufgabe der zukünftigen Musikwissenschaft sein, das Chaos der Möglichkeiten zu sichten, durch klar erkannte Gesetze eine Ordnung zu schaffen, durch welche allein die höhere Freiheit der Kunst errungen wird...]

recognizes the problem, he apparently does not see the role of equal temperament in solving it. Chords are not discrete in his system—they are linked transformationally—but transformations are only a means of navigating the "chaos"; they are blind to normative progressions underlying it, and therefore to the various meanings a chord may have depending on its context.

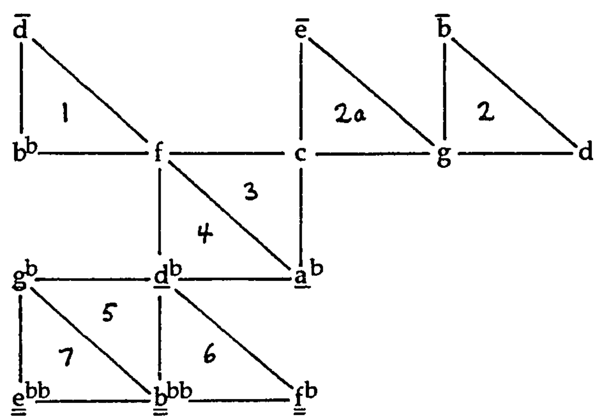
Whenever Oettingen encounters enharmonic change in music—he avoids music analysis, so this is rare—his system breaks down. The abrupt appearance of a "new" chord that occupies a remote position on the Table has no rationale in the transformational model. Instead of salvaging his system, however, Oettingen faults the music in such passages for shifting incongruously into the "wrong" key. *Modulationsfehler* occur quite often, and even in the work of great composers. Oettingen cites several examples in Beethoven, one of which is given below in Example 2-9. The excerpt is from the development section of Beethoven's Piano Sonata in C, op. 2, no. 3, and the modulation error occurs at measure 103.

EXAMPLE 2-9: "MODULATION ERROR" IN OP. 2, NO. 3



The passage in question begins in m. 97 on the dominant of E^b major, and arrives in m. 109 on a tonicized D major chord. We have numbered the main chords of the passage for ease of reference. The "modulation error" occurs when the F and A^b of m. 102 are reinterpreted as the third and fifth of a C$^\#$ major chord in m. 103. According to Oettingen, the chord at m. 103 should be '$\underline{d}^b$–f–$\underline{a}^b$', not $\bar{c}^\#$–$\bar{e}^\#$–$\bar{g}^\#$, because this chord is directly related to the 'f–$\underline{a}^b$–c' of m. 102 through the *Leittonschritt*.[32] He criticizes Beethoven for leading the passage into D major when, transformationally speaking, the true goal is E^{bb} major. Oettingen's view of the passage is most easily expressed in terms of the Table.

EXAMPLE 2-10: TRANSFORMATIONAL ANALYSIS OF OP. 2, NO. 3 (MM. 97–109)



Transforms

1—2: 3Q

2—3: Q + W

3—4: L

4—5: W

5—6: T

6—7: Q

Chord 2 in Example 2-10 stands for the diminished seventh $\bar{b}$–d–f–$\underline{a}^b$, and is three *Quintschritte* (3Q) away from chord 1. The relation between chords 2 and 3 is indirect, but one way of connecting them is through a

32 Oettingen, 143. There is a misprint in Oettingen's text, which confusingly equates $\underline{d}^b$–$\underline{\underline{f}}^b$–$\underline{a}^b$ with Beethoven's $\bar{c}^\#$–$\bar{\bar{e}}^\#$–$\bar{g}^\#$. It is clear from the context that Oettingen means f-natural rather than $\underline{\underline{f}}^b$.

combination *Quintschritt* and *Wechsel* (Q+W). This possibility is shown by a hypothetical move from chord 2a—which does not occur in the music—to chord 3. The enharmonic change occurs between chords 3 and 4. In Example 2-10, however, the two chords are directly related through L. Chords 4 and 5 are related through W, chords 5 and 6 through T, and chords 6 and 7 through Q.

Oettingen's analysis of these few measures is mechanistic and leaves a cold impression. Chords seem little more than passive objects in the Beethoven excerpt; one may be transformed into another but none possesses the dynamic quality we experience when listening to this music. Because Oettingen neglects the propulsive aspect of harmony, his transformations are a better model of chord connection than chord progression. To show the progressive nature of chords, he needs a theory of function. Without this he cannot acknowledge the ambiguity of $\bar{c}^{\#}$–$\bar{\bar{e}}^{\#}$–$\bar{g}^{\#}$ at m. 103, or interpret D major at m. 105 in the context of the whole movement. The C$^{\#}$ major chord is a sudden and irrational incursion into his system. He must read it either as $\underline{d}^{b}$–f–$\underline{a}^{b}$, and fault Beethoven, or revise his theory. The exasperating thing is that Oettingen does sense the need for a higher criterion—one that will control chordal transformation and reduce the "chaos of possibilities"—but instead of yielding his theory to this need he forecasts darkly that a "functional connection of these forces might be very hard to find."[33] A theory of harmonic function was not as far off as Oettingen imagined.

[33] Oettingen, 156.

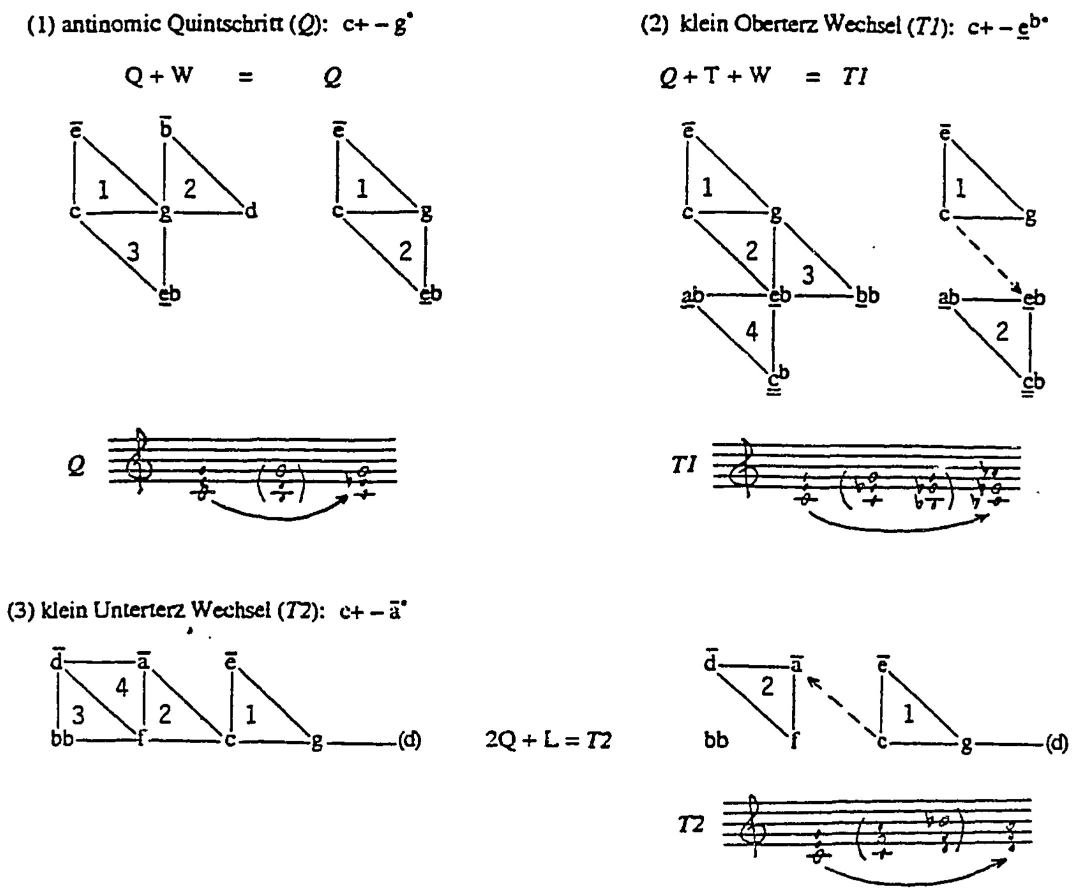
2.9 Tone-Systems Generalized

Oettingen concludes chapter 3 of *Harmoniesystem* with a discussion of distantly related tone-systems, a discussion that becomes possible once he has explained chordal connection. Chapter 4 organizes close and distant relations in a series of four symmetrical charts. As before, the proximity between systems is determined by the presence of parallel or reciprocal chords. We have seen the Table used to clarify pitch and chord relations in Oettingen's system. In chapter 4, it is used to summarize tone-system relations. The Table is by modern standards a complete pitch space, since it may be used to model tone, chord, and key relations.

Oettingen applies the standard of smooth chord connection to moves between tone-systems. Remote systems must be mediated through systems whose tonic (or phonic) chord is directly related to both of them. This chord need not be a pivot chord in the traditional sense. In a move from tonic 'c' to tonic '$\underline{d}^{b}$', for example, f–$\underline{a}^{b}$–c serves as mediator because it is both W of 'c' and L of '$\underline{d}^{b}$'. On the other hand, $\bar{e}$–g–$\bar{b}$ serves as mediator between tonic 'c' and tonic '$\bar{b}$' because it is L of 'c' and W of '$\bar{b}$'. Eight systems can be obtained either directly from tonic 'c', or at most through the mediation of its antinomic *Wechsel*. The direct systems are phonic '$\bar{e}$', phonic '$\bar{b}$', tonic 'f', and tonic 'g' (see Example 2-6). The W-mediated systems are tonic '$\underline{a}^{b}$', tonic '$\underline{d}^{b}$', phonic 'g', and phonic 'f'. In each of these systems f–$\underline{a}^{b}$–c (W of 'c') is the mediator. Through the application of W to the tonics and phonics of these systems, Oettingen obtains four more W-mediated systems: tonic '$\bar{e}$' (W of phonic '$\bar{e}$'), tonic '$\bar{b}$' (W of phonic '$\bar{b}$'), phonic '$\underline{a}^{b}$' (W of tonic '$\underline{a}^{b}$'), and phonic '$\underline{d}^{b}$' (W of tonic '$\underline{d}^{b}$').

The W-mediated systems inspire several shortcut transformations whose purpose is to circumvent combinations of W, Q, and T. Using the basic transformations, a move from tonic 'c' to phonic 'g' is indirect and requires at least one intervening step (c–e–g [Q] g–b–d [W] c–$\underline{e}^{b}$–g). Even the combination Q + W, however, is not simple or clear enough to satisfy Oettingen, so he introduces the shortcut antinomic *Quintschritt* (*Q*). Although Q is always more intelligible than *Q*, there are many cases where Oettingen's shortcut reflects the harmonic situation better than Q + W. Two other shortcut transformations are the *klein Oberterz Wechsel* (*T1*) and *klein Unterterz Wechsel* (*T2*), both of which are similar to T but involve a minor third instead of a major third. Example 2-11 illustrates *Q*, *T1*, and *T2*. Shortcut transformations are Oettingen's response to commonplace harmonic relations that happen to be remote in his theory; in most situations he prefers the simplicity of Q, T and W. Riemann later adopts both basic and shortcut transformations into his function theory.

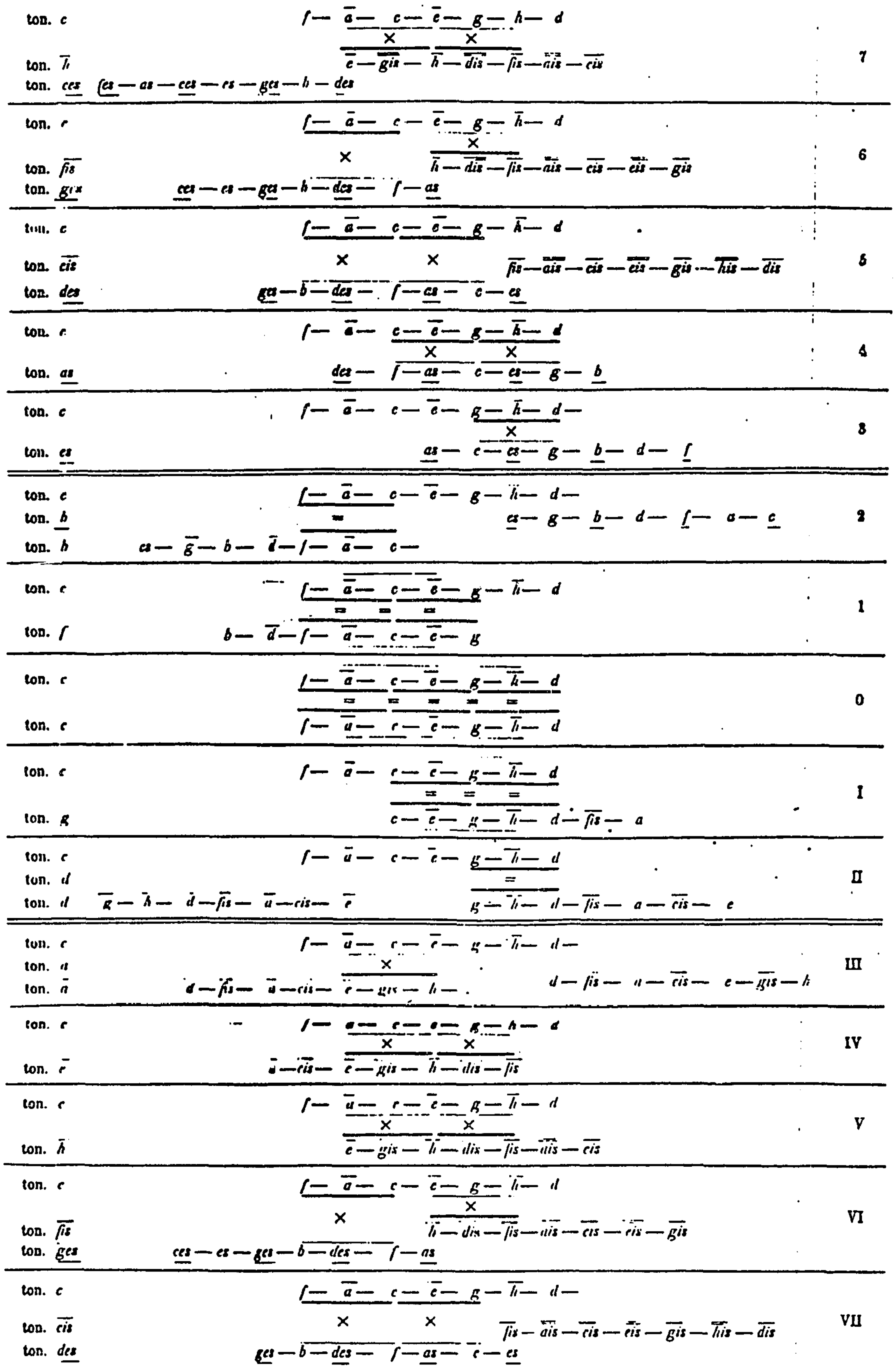
EXAMPLE 2-11: SHORTCUT TRANSFORMATIONS



The fourth chapter of *Harmoniesystem* is entitled "Verwandtschaftskreis der reinen Tongeschlechter" (The Circle of Relation of Pure Tone-Systems). Here Oettingen compiles a list of tonic and phonic systems that bear some relation to tonic 'c' or phonic 'ē', and presents his results in a series of four tables. Table A shows the relation of tonic systems to tonic 'c'. Table B, its dual, shows the relation of phonic systems to phonic 'ē'. Table C compares tonic 'c' with the antinomic phonic systems, and Table D compares phonic 'ē' with the antinomic tonic systems. Tone-systems may be related in one of two ways: If they possess identical chords, their relation is parallel. If they possess opposite chords, their relation is reciprocal. For any given system, there are twelve homonomically related systems and twelve antinomically related systems.

Example 2-12 reproduces Table A, which shows the "Tonic-to-Tonic Relationship" (*Tonisch-tonische Verwandschaft*) with tonic 'c' serving as the reference point. The closest and most trivial relation to tonic 'c' occurs at 0, the table's midpoint, where tonic 'c' is compared with itself. The next closest relations are at 1 and I, where Oettingen compares tonic 'c' with tonics 'f' and 'g'. Tonic-tonic relations become more remote as Oettingen cycles through the series of fifths. If two systems possess identical chords, the sign '=' denotes parallel relation and shows the chords that are common to both systems. If two systems possess opposite (*gegensätzlich*) chords, the sign 'x' denotes reciprocal relation and shows the chords that are opposites. Oettingen sometimes calls reciprocal systems *Wechsel* systems. Arabic numbers 1–7 of Table A compare fifth-related tonic systems on the *Unterdominante* side of tonic 'c'; Roman numerals I–VII compare fifth-related tonic systems on the *Oberdominante* side. Oettingen uses heavy and light horizontal lines to underscore the significance of chords to their systems. Heavy lines indicate *Hauptklänge* and light lines indicate *Nebenklänge*. In tonic systems, *Hauptklänge* are major chords and *Nebenklänge* are minor chords. (The opposite holds in phonic systems, where the *Phonica* and its *Regnante* are minor and the antinomic *Unterterz* and *Leitklang* are major).

EXAMPLE 2-12: OETTINGEN'S TABLE OF TONIC-TONIC RELATIONSHIP



At 1/I of Example 2-12 there are three parallel chords (two *Hauptklänge* and one *Nebenklang*). At 2/II there is just one parallel chord. The double lines between 2/3 and II/III are the boundary at which parallel relations give way to reciprocal ones. At 3/III tonic 'c' shares no identical chords with tonics '$\underline{e}^{\flat}$' and '$\overline{a}$', but there is a reciprocal relation in each case. The systems 3–7 and III–VII all possess reciprocal relations with tonic 'c', and are therefore *Wechsel* systems. Reciprocal relations initially increase as one moves away from the central system, but then diminish—4/IV and 5/V each have two reciprocal chords, but 6/VI has just one and 7/VII has none at all. Oettingen establishes new reciprocal connections at 6/VI and 7/VII, however, through enharmonic change. By changing '$\underline{\underline{g}}^{\flat}$' to '$\overline{\overline{f}}^{\#}$' at 6 and '$\underline{c}^{\flat}$' to '$\overline{b}$' at 7, he opens up tone-systems that bear a relation to tonic 'c'. Enharmonic relations at VI and VII open up analogous systems. There are two points to be made about these enharmonically related systems. First, they are acoustically similar but not equivalent. Second, by invoking them Oettingen closes the cycle of homonomic systems related to tonic 'c'—through enharmonic change, the systems considered at 5, 6, and 7 are identical to those at VII, VI, and V. By closing the cycle, Oettingen restricts the "chaos of possibilities" to what is at least a manageable system of systems. All together, Table A shows that twelve homonomic systems are related to tonic 'c': Five are parallel systems (Oettingen counts the self-identity relation at 0), and seven are reciprocal or *Wechsel* systems.

Several observations may be made from Table A, which will save us from discussing Table B and simplify our discussion of Tables C and D:

1) Parallel relations are stronger than *Wechsel* relations between homonomic systems (whether the tonic-tonic systems of Table A or the phonic-phonic systems of Table B).

2) No systems can be related through both parallel and reciprocal chords.

3) A *Hauptklang* remains a *Hauptklang* between parallel systems—the *Tonica* of 'c' is the *Unterdominante* of 'g' and the *Oberdominante* of 'f'; Although the *Oberdominante* and *Unterdominante* systems are equidistant from tonic 'c', tonic 'f' is closer than tonic 'g' to 'c'.

4) A *Nebenklang* becomes a *Hauptklang* in *Wechsel* systems and vice-versa. This shift of significance is one reason that *Wechsel* systems are more remotely related than parallel systems to the central tonic or phonic system.

5) Whenever *Hauptklänge* become *Nebenklänge* the relation between homonomic systems is stronger than when the opposite occurs. The *Wechsel* systems at 3–7 are therefore more closely related to tonic 'c' than those at III–VII.

Like Tables A and B, Tables C and D are also symmetrically related. Since Oettingen is comparing antinomic systems in these tables, some of the above observations must be reversed here:

1) *Wechsel* relations are stronger than parallel relations between antinomic systems (whether the tonic-phonic systems of Table C, or the phonic-tonic systems of Table D).

2) A *Hauptklang* remains a *Hauptklang* between *Wechsel* systems. Reciprocal relations are therefore much stronger between antinomic systems than between homonomic systems.

3) A *Nebenklang* becomes a *Hauptklang* in parallel systems and vice-versa.

A finite "field of pitches" (*Bereich der Töne*) emerges from the tone-system relations expounded in Tables A–D. This field represents a sort of aggregate collection, consisting of all pitches belonging to the systems related to tonic 'c' and phonic '$\overline{\text{e}}$', and is presented by Oettingen in the form of the Table. Example 2-13 presents Oettingen's field of pitches in tabular form. The field is a closed system and in this respect is unlike the Table in Example 2-5. No pitch outside its boundaries bears a relation to tonic 'c' or phonic '$\overline{\text{e}}$'; indeed, those within its first and last columns occur only in systems with the most tenuous relation to tonic 'c' and phonic '$\overline{\text{e}}$'. The field effectively shrinks the Table to manageable size, yet there are still many enharmonic relations among its elements. Each series ends with a pitch enharmonically related to the first pitch of the next series, and the entire top series is enharmonically related to the bottom series. In much music the field would have to be considerably widened to include the overlapping fields of other tone-systems.

EXAMPLE 2-13: OETTINGEN'S "FIELD OF TONES"

Bereich der Töne.

$\overline{\overline{\text{e}}}$	$\overline{\overline{\text{h}}}$	$\overline{\overline{\text{fis}}}$	$\overline{\overline{\text{cis}}}$	$\overline{\overline{\text{gis}}}$	$\overline{\overline{\text{dis}}}$	$\overline{\overline{\text{ais}}}$	$\overline{\overline{\text{eis}}}$	$\overline{\overline{\text{his}}}$
$\overline{\text{c}}$	$\overline{\text{g}}$	$\overline{\text{d}}$	$\overline{\text{a}}$	$\overline{\text{e}}$	$\overline{\text{h}}$	$\overline{\text{fis}}$	$\overline{\text{cis}}$	$\overline{\text{gis}}$
as	es	b	f	c	g	d	a	e
$\underline{\text{fes}}$	$\underline{\text{ces}}$	$\underline{\text{ges}}$	$\underline{\text{des}}$	$\underline{\text{as}}$	$\underline{\text{es}}$	$\underline{\text{b}}$	$\underline{\text{f}}$	$\underline{\text{c}}$

Example 2-14 segments the tonal field to show which systems are related to tonic 'c' and which are related to phonic 'ē'. Here letter names refer not to specific pitches, but to entire tone-systems. Together the four segments synopsize the information of Tables A–D in an easy-to-read, if less detailed, format. The upper left and right segments abbreviate Tables A and B, and Tables C and D are represented by the lower left and right segments. In each segment, the central system is highlighted among its homonomic or antinomic relations. Bracketed systems stand at what Oettingen calls the "limit of direct relation," possessing common tones with each of the central systems, but no parallel or reciprocal chords.

EXAMPLE 2-14: SEGMENTING THE "FIELD OF TONES"

Homonome Systeme:

Tonisch *c* mit Tonisch:

	$(\bar{d})$	$\bar{a}$	e	$\bar{h}$	$\overline{fis}$	(cis)
(es)	b	f	**c**	g	d	(a)
(ces)	$\underline{ges}$	$\underline{des}$	$\underline{as}$	$\underline{es}$	$(\underline{b})$	$(\underline{f})$

Phonisch $\bar{e}$ mit Phonisch:

$(\bar{h})$	$\overline{fis}$	$\overline{cis}$	$\overline{gis}$	$\overline{dis}$	$\overline{ais}$	
$\bar{g}$	$\bar{d}$	$\bar{a}$	**$\bar{e}$**	$\bar{h}$	$\overline{fis}$	$(\overline{cis})$
(es)	b	f	c	g	(d)	

Antinome Systeme:

Tonisch *c* mit Phonisch:

	$(\overline{fis})$	$(\overline{cis})$	$\overline{gis}$	$\overline{dis}$	$(\overline{ais})$	
	$(\bar{d})$	$\bar{a}$	$\bar{e}$	$\bar{h}$	$\overline{fis}$	$(\overline{cis})$
(es)	b	f	**c**	g	d	(a)

Phonisch $\bar{e}$ mit Tonisch:

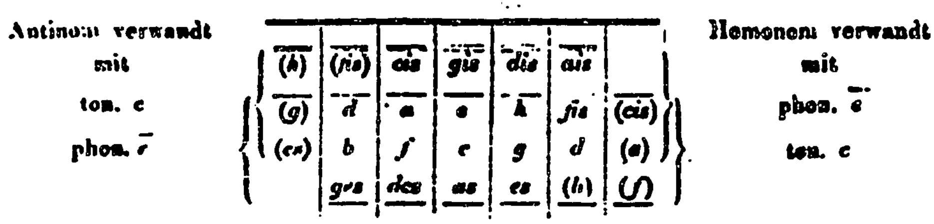
$(\bar{g})$	$\bar{d}$	$\bar{a}$	**$\bar{e}$**	$\bar{h}$	$\overline{fis}$	$(\overline{cis})$
(es)	b	f	c	g	(d)	
$(\underline{ces})$	$\underline{ges}$	$\underline{des}$	$\underline{as}$	$(\underline{es})$	$(\underline{b})$	

The four segments of Example 2-14 overlap and may be presented as one comprehensive table. Oettingen presents such a table so that readers may convince themselves of "the perfect symmetry that is operational here."[34] Because this table, shown in Example 2-15, represents

[34] Oettingen, 176.

homonomic and antinomic relations, some elements will have as many as four meanings. For example, 'g' may be understood as antinomic to tonic 'c', homonomic to phonic 'ē', antinomic to phonic 'ē', or homonomic to tonic 'c'. The larger brackets on either side of the table express these possibilities. Parentheses are used as before to represent the tenuously related common-tone systems.

EXAMPLE 2-15: COMPREHENSIVE TABLE OF TONE-SYSTEM RELATIONS



Antinom verwandt mit	$(\overline{h})$	$(\overline{fis})$	$\overline{cis}$	$\overline{gis}$	$\overline{dis}$	$\overline{ais}$		Homonom verwandt mit
ton. c	$(\overline{g})$	$\overline{d}$	$\overline{a}$	$\overline{e}$	$\overline{h}$	fis	$(\overline{cis})$	phon. $\overline{e}$
phon. $\overline{e}$	(es)	b	f	c	g	d	(e)	ton. c
		ges	des	as	es	(h)	(f)	

Example 2-15 does not show the parallel and reciprocal connections that bind related tone-systems to tonic 'c' and phonic 'ē'. Tables A–D show such connections but, unlike Example 2-15, must represent homonomic and antinomic relations separately. Example 2-16 preserves the details of parallel and reciprocal relation between tonic systems, but in a format slightly different from the one given in Example 2-12. Tonic 'd' is the central system in Example 2-16, and its five consonant chords are listed in the top horizontal series. *Hauptklänge* are given in large type ('g', 'd', and 'a'), and *Nebenklänge* in smaller type (*fis* = b̄–d–f̄#, and *cis* = f̄#–a–c̄#). Example 2-16 provides a bird's-eye view of homonomic systems related to tonic 'd', but shows also the manner of relation. The columns beneath the five tonic 'd' chords each list five homonomic systems in which the

Hauptklänge occur, identically (||) or reciprocally (×). The significance of the chord to the related system is given in the far left column. When one of the five chords appears as a subsidiary form (*Nebengebilde*) in a related system, the name of that system appears in small type. The first column of Example 2-16 would be read as follows: The chord g–$\bar{b}$–d appears identically in tonic 'd' as *Unterdominante,* reciprocally in tonic '$\underline{e}^{\flat}$' as *Terzklang,* identically in tonic 'g' as *Tonica,* reciprocally in '$\underline{a}^{\flat}$' as *Leitklang,* and identically in tonic 'c' as *Oberdominante.* Notice that the *Hauptklänge* of tonic 'd' each occur identically in three homonomic systems, and reciprocally in two homonomic systems. The *Nebenklänge* each occur reciprocally in three homonomic systems, and identically in two homonomic systems.[35] Oettingen gives analogues of Example 2-16 to Tables B, C, and D, but with tonic 'c' and phonic '$\bar{e}$' as the central systems.

EXAMPLE 2-16: HOMONOMIC SYSTEMS RELATED TO TONIC 'D'

Homonome Verwandtschaft.

Klänge von ton. *d*:	*g*	*fis*	*d*	*cis*	*a*	Die Tonarten der verschiedenen Horizontalreihen sind Klänge von:	
Unterdom.-Kl.	*d* ||	*cis* ×	*a* ||	*gis*	*e* ||	ton. *a*	Klänge von phonisch *d*
Terz-Klang	*es* ×	*d* ||	*b* ||	*a* ||	*f* ×	ton. *b*	
Tonica-Klang	*g* ||	*fis* ×	*d* ||	*cis* ×	*a* ||	ton. *d*	
Leit-Klang	*as* ×	*g* ||	*es* ||	*d* ||	*h* ×	ton. *es*	
Dominant-Kl.	*c* ||	*h* ×	*g* ||	*fis* ×	*d* ||	ton. *g*	
Die Tonarten der verschiedenen Vertikalreihen sind Klänge von:	ph. *g*	ph. *fis*	ph. *d*	ph. *cis*	ph. *a*		
	Klänge von tonisch *d*						

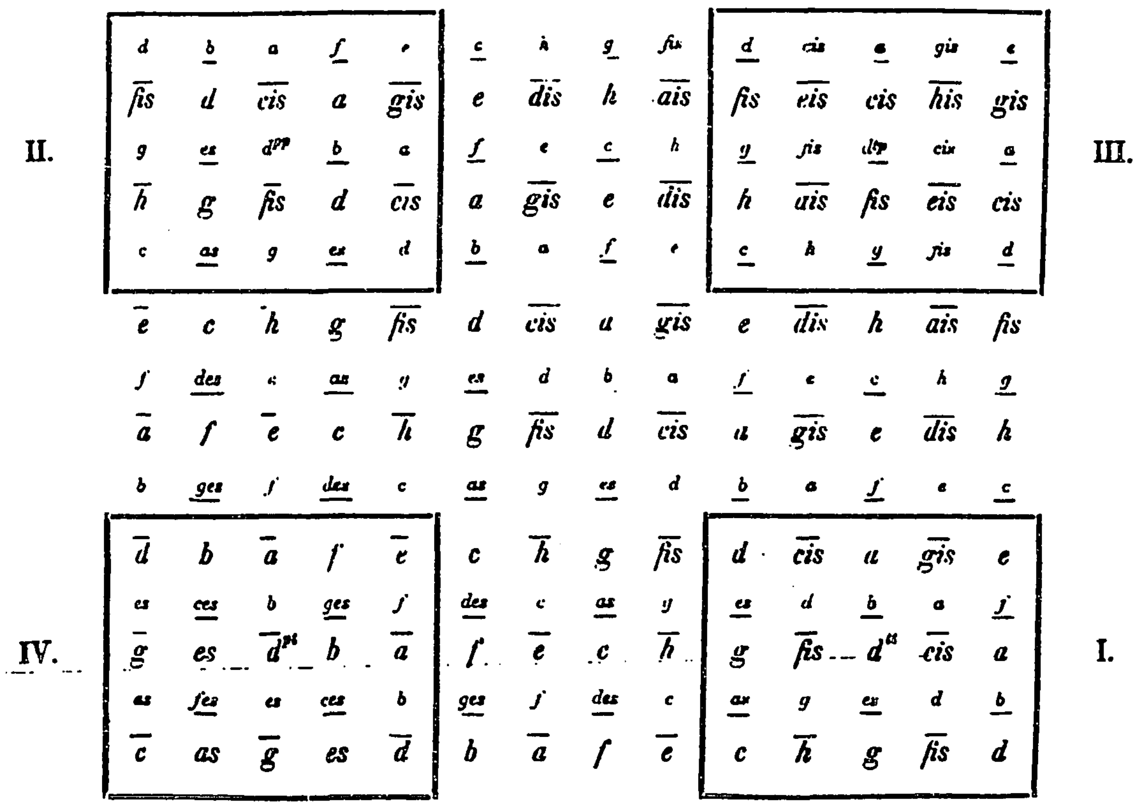
35 There are some errors in Example 2-16: Chords $\underline{b}$ and $\underline{es}$ in column 3 are reciprocal (×) relations, as is the unmarked chord gis in column 4.

The tone-systems represented in Example 2-16 belong to three different fifth-series. If one imagines a Table of Relations centered on pitch 'd', these series are:

1) the central series (...g–d–a...)
2) the upper third series (...$\bar{b}$–$\bar{f}^{\#}$–$\bar{c}^{\#}$..)
3) the lower-third series (...$\underline{e}$b–$\underline{b}^{b}$–$\underline{f}$..)

These three fifth-series are interlocked in Example 2-16. In rows 1, 3, and 5, elements of the central and upper-third series alternate. In rows 2 and 4 elements of the central and lower-third series alternate. Oettingen concludes his discussion of tone-system relations by extending the interlocking series of Example 2-16 horizontally and vertically, to yield one large Table that captures the homonomic and antinomic relations of Tables A–D. Tonic and phonic 'd' are again the central systems. This table is reproduced in Example 2-17. Section I shows the relation of all tonic systems with tonic 'd^{tt}' (tt = tonic-tonic). Section II shows the relation of all phonic systems with phonic 'd^{pp}' (pp = phonic-phonic). Sections III and IV show the relation of phonic systems with tonic 'd^{tp}' and tonic systems with phonic 'd^{pt}' (tp = tonic-phonic; pt = phonic-tonic).

EXAMPLE 2-17: TABULAR SUMMARY OF TONE-SYSTEM RELATIONS



2.10 Summary of Tone-System Relations

The relation between tone-systems is determined by parallel and reciprocal chords. When one or more chords occur identically in two systems, the systems are called parallel systems. When one or more chords occur oppositely in two systems, the systems are called *Wechsel* systems. The degree of relation between parallel or *Wechsel* systems depends on both the number of parallel and reciprocal chords, and their significance as either *Hauptklänge* or *Nebenklänge*. In tonic systems the *Hauptklänge* are *Tonica, Oberdominante,* and *Unterdominante*; the

Nebenklänge are the *Leitklang* and *Terzklang*. In phonic systems the *Hauptklänge* are *Phonica, Unterregnante*, and *Oberregnante*; the *Nebenklänge* are the *Leitklang* and *Unterterzklang*. These terms describe chordal derivation, not function: Oettingen does not present a tonal syntax to regulate the progression of *Hauptklänge* and *Nebenklänge*. He presents a theory of chordal connection instead, where chords are transformed into adjacent chords on the Table through transposition (homonomic *Quintschritt*), inversion (antinomic *Wechsel*), or a combination of both (antinomic *Terzschritt* and *Leittonschritt*). The path between remote (nonadjacent) chords is mediated through series of transformations; the gap between remote tone-systems (systems sharing no adjacent chords) is bridged by parallel- or *Wechsel*-related systems.

Parallel relations bind homonomic systems most intimately. The closest homonomic relations are those in which *Hauptklänge* of the starting system remain *Hauptklänge*, as in systems 1/I and 2/II of Example 2-12. The next closest relations are those in which *Hauptklänge* of the starting system become *Nebenklänge*, as in systems 3 and 4 (but not III and IV) of Example 2-12. In antinomic systems, however, reciprocal relations determine intimacy. The closest relations are those in which *Hauptklänge* of the starting system are reciprocal to *Hauptklänge* of the related system. Next closest are those in which *Hauptklänge* of the starting system are reciprocal to *Nebenklänge* of the related system. A weaker relation obtains when *Nebenklänge* of the starting system are *Hauptklänge* in the related system, and the relation is even weaker when *Nebenklänge* remain *Nebenklänge*. The most tenuous relation is the common-tone relation, where homonomic or antinomic systems have one or two tones in common but no parallel or reciprocal connections.

Example 2-15 does not make explicit the relation between tonic 'c', tonic 'g', and phonic 'ē'. Is 'g' closer to 'c', or is phonic 'ē'? Tonic 'g' is in fact closer, though it has fewer chords in common with 'c', because the *Hauptklänge* of 'c' remain *Hauptklänge* in 'g' (see Example 2-12) but become *Nebenklänge* in phonic 'ē'. On the other hand, Example 2-15 clarifies relations that are obscure in Tables A–D. Since the four tables cycle mechanically through series of fifths, the relation of enharmonic systems to the central system is always somewhat unclear. In Example 2-12, for example, the implied relation of tonic 'c' to tonic 'b̄' through a series of five fifths (tonic 'b̄' is system V) hides the much closer relation of these systems through the *Leitklang* of 'c', whose reciprocal is the *Tonica* of 'b̄'. This directness is captured nicely in Example 2-15, as are the relations of tonics 'ā', 'ē', 'd̲♭', 'a̲♭', and 'e̲♭' to tonic 'c'. Oettingen considered it a great theoretical advance to distinguish fifth relations from third relations—'e' from 'ē'—a distinction that remained hidden in theories based on equal temperament. In *Harmoniesystem*, third-relations received their due and the whole notion of direct harmonic relationship was broadened to include chords and tone-systems that occupied fairly remote positions on the circle-of-fifths. Oettingen was convinced that in new music as well as old, modulation had always been based on parallel connections between fifth-related systems, and reciprocal connections between third-related systems. The Table of Relations allowed him to express both dimensions of his dualistic harmonic conception.

CHAPTER 3: MUSICAL LOGIC AND MUSICAL SYNTAX: TWO PARADIGMS OF HARMONIC FUNCTION

3.1 Introduction

Riemann's most enduring contribution to music theory was his theory of harmonic functions. This took many years to work out, and was unfinished in some respects when he died, but in one sense or another harmonic function stood behind all Riemann's inquiries into the tonal structure of music. The goal of this chapter is to define two senses in which harmonic function informed Riemann's work, and to establish a connection between one of these and the Table of Relations. The two senses of harmonic function we shall call "categorical" and "chordal." The categorical sense was associated with Riemann's concept of musical logic, and had as its paradigm a chord progression known as the *große Cadenz*. The chordal sense was associated with Riemann's concept of musical syntax, and had as its paradigm the Table of Relations. Categorical function distinguishes between chords and chordal function; a tonic function, in the categorical sense, is not the same as a tonic chord. Chordal function makes no such distinction; tonic chords and functions are for practical purposes identical. Function-as-category and function-as-chord were often confused by Riemann, but we shall claim there was a gradual shift away from the *große Cadenz* and toward the Table of Relations in Riemann's early period.[1]

[1] See Elmar Seidel, "Die Harmonielehre Hugo Riemanns," in *Beiträge zur Musiktheorie des 19. Jahrhunderts*, ed. Martin Vogel (Regensburg: Gustav Bosse Verlag, 1966), 43–61. Seidel divides Riemann's career into early (1872-77), middle (1877–1909), and late (1909–19) periods. We depart slightly from this scheme by considering the essay "Die Natur der

Three early works will serve our goal: the predissertation essay "Musikalische Logik" (1872); the dissertation "Ueber das musikalischen Hören" (1873);[2] and the second published treatise *Musikalische Syntaxis* (1877).[3] "Die Natur der Harmonik" (1882) will be treated briefly, and *Skizze einer neuen Methode der Harmonielehre* (1880), the sole practical work of Riemann's first period, will be deferred to Chapter 5. Though we are concerned with speculative theory in this chapter, we should stress that speculative and practical issues are not always easy to separate in Riemann. Refinements made to the Table in Riemann's pedagogical works grew out of early speculative work and, in turn, spawned Riemann's later speculations in "Ideen zur einer Lehre von den Tonvorstellungen" (1914/15).

Harmonik" (1882) among Riemann's early works, which were generally more speculative than pedagogical in outlook.

[2] This work has a complicated history: It was rejected as a dissertation at the University of Leipzig by M. W. Drobisch and Oscar Paul, but accepted at Göttingen under the title "Ueber das musikalischen Hören" (Ph.D. diss., University of Göttingen, 1873). The work was published under this title by Vandenhoeck & Ruprecht (Göttingen, 1874) and F. Andrä (Leipzig , 1874); and by C. F. Kahnt (Leipzig, 1874) as *Musikalische Logik: Hauptzüge der physiologischen und psychologischen Begründung unseres Musiksystems.* Riemann incorporated parts of the essays "Musikalische Logik: Ein Beitrag zur Theorie der Musik" (1872) and "Ueber Tonalität" (1872) into the dissertation (see Chap. 2, n. 5).

[3] Riemann, *Musikalische Syntaxis: Grundriß einer harmonischen Satzbildungslehre* (Leipzig: Breitkopf und Härtel, 1877). Riemann's "Neue Schule der Harmonik" and "Musikalische Grammatik" were completed between 1874 and 1877. Both were conceived as pedagogical counterparts to the earlier works, but neither was published and the manuscripts were eventually destroyed.

3.2 An Overview of the Early Works

Riemann's early works are not usually thought of in connection with harmonic function. All of them antedated the *Funktionbezeichnungen* (function symbols) introduced in *Vereinfachte Harmonielehre* (1893), yet all were decidedly caught up in issues of functionality. Whether these issues concerned the logical meanings of chords, their syntactical connections, or their psychological *Vorstellungen* (representations), Riemann was grappling with problems of function and working out solutions that heralded his mature function theory.

Of the early works, the Table of Relations appeared only in "Ueber das musikalische Hören" and "Die Natur der Harmonik." It was presented in a footnote in the dissertation, and treated superficially in comparison with the earlier treatment by Oettingen. Riemann's commentary suggested a nominal acceptance of just intonation, not because it was simpler or more pure, but because it underscored functional distinctions between pitches with the same letter name. The symmetry of the Table possessed more obvious appeal, since it captured the dualistic relations that ensued from his "kleine Hypothese" of harmonic undertones. Riemann focused, however, on pitch relations and largely ignored the "äussere Dualität" of chordal and key relations. If the Table seemed a mark of pedigree at this stage—Riemann's way of saying he was *au courant* with recent music theory—this assessment became harder and harder to maintain. By the end of Riemann's early period, one's sense is that the Table was no ornament in "Ueber das musikalische Hören," but a harbinger of theoretical change.

Riemann moved forward with greater independence after the dissertation. He continued to acknowledge the work of Rameau, Hauptmann, Helmholtz, and Oettingen, but did not persist in his earlier attempts to reconcile the approaches of these theorists. He began to withdraw in particular from Helmholtz and Hauptmann—the most esteemed of his *Vorfahren*—and to ally himself with Oettingen, to whom he dedicated *Musikalische Syntaxis.* The significance of this dedication must be qualified, for Riemann omitted the Table from *Musikalische Syntaxis* and rejected both the phonic system and just intonation.[4] It was Oettingen's theory of chord connection that interested him. This offered a concrete yet flexible means for relating chords, and was more practical than the dialectical system proposed by Hauptmann. For better (and for worse), Riemann initiated a process in the late 1870s that led to a chordal conception of harmonic function (see Ex. 3-1 below). The shift from function-as-category to function-as-chord was underway.

When the Table resurfaced in "Die Natur der Harmonik" it modeled not only Riemann's dualism, but also the syntactical view of chord relation expounded in *Musikalische Syntaxis.* It was reinvented to conform with other changes as well, including a revised view of harmonic third relations and an explicitly psychological conception of tonality. Riemann's relation of tonality to the Table invested it with the directional force that was lacking in Oettingen's earlier treatment.

[4] Riemann, *Musikalische Syntaxis,* viii. Concerning intonation, Riemann writes: "Pure tuning is unnecessary, since interpretation actually accommodates itself excellently to tempered relations." [Nöthig ist die reine Stimmung nicht, da sich faktisch unsere Auffassung mit den temperirten Verhältnissen vortrefflich abfindet.]

3.3 Mapping Dualism and Function: The *Tonnetz*

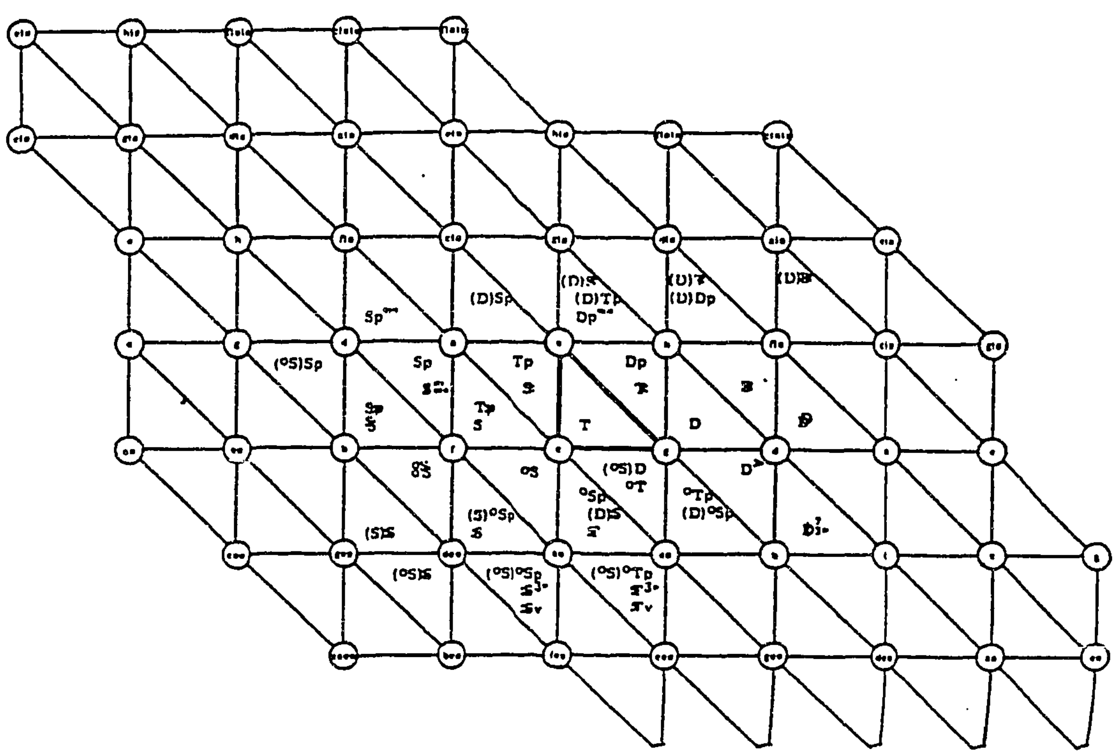
Our claim that the Table modeled both dualism and function contradicts recent scholarship, which has maintained these were opposed rather than complementary principles for Riemann.[5] The connection between dualism and function should come as no surprise, since the basic moves within Oettingen's dualist-transformational model—moves intimately connected with the Table—supplied the harmonic relations at the core of Riemann's function theory. Perhaps the connection has been overlooked because the Table itself has been overlooked, or misread to show inconsistency where none exists. Part of the blame belongs to Riemann for not spelling out the Table's relation to harmonic function, and for failing to distinguish the chordal aspects from the categorical ones.

In claiming that the Table is related to chordal harmonic function, we shall want to claim that Riemann's harmonic conception was more transformational than *Vereinfachte Harmonielehre* and later works indicate. There was tension between the concepts of dualism and function, and the Table did not convey functional-hierarchical relationships as accurately as dualistic ones, but Riemann clearly intended it to model both aspects of his theory. The evidence for this lies chiefly in his development of *Funktionsbezeichnungen* that were congruent with the Table's geometry. Renate Imig has characterized the Table as a *Tonnetz* (tonal network), and shown the intimate relationship between its structure and Riemann's function symbols. We reproduce Imig's

[5] See David Bernstein, "Schoenberg Contra Riemann: *Stufen*, Regions, *Verwandschaft*, and the Theory of Tonal Function," *Theoria* 6 (1992): 23–53.

Tonnetz in Example 3-1 so that readers may gain an immediate sense of the correspondence between harmonic function and the Table.[6]

EXAMPLE 3-1: IMIG'S *TONNETZ*



We assume our readers will understand at least the basic relations in Example 3-1. The nearness of relation between *Hauptklänge* (T, S, and D),

[6] Renate Imig, *Systeme der Funktionsbezeichnung in den Harmonielehre seit Hugo Riemann,* Orpheus-Schriftenreihe zu Grundfragen der Musik, vol. 9 (Düsseldorf: Gesellschaft zur Förderung der systematischen Musikwissenschaft, 1970), 258.

Hauptklänge and their *Parallele* (T–Tp; S–Sp; D–Dp), and *Hauptklänge* and their *Leittonwechselklänge* (T–<T; S–<S; D–<D), is given visual form within a structure that also shows the dualistic relation between major and minor chords: Function and dualism coexist in a complementary manner that is clarified rather than obscured by the Table.[7] To be true to the Table, Riemann should have posited *Hauptklänge* corresponding to A^b major and E major; these chords stand in the same geometrical relation to C as do the subdominant and dominant. Riemann should have acknowledged direct third relations, in other words, just as he acknowledged direct fifth relations. Later he did consider expanding the *Funktionbezeichnungen* to include such relations, proposing the symbol 3+ for the major *Hauptklang* built on the mediant, and III+ for the major *Hauptklang* built on the minor submediant. The addition of fourth and fifth "harmonic pillars" would have placed the Table's vertical relations on a par with its horizontal ones. Unfortunately, Riemann died before he could assimilate these ideas into his theory.

The relation between the Table and music psychology is worth mentioning here. Riemann placed much stock in what he called the "tonal imagination" of listeners, and *Tonvorstellung* was the backbone of his harmonic theory by the early 1880s. To the extent that the Table modeled this theory, one may assume it modeled aspects of Riemann's music psychology as well. Chapter 4 examines the *Vorstellung* concept in

[7] For typographical convenience, we write *Wechsel* signs '<' and '>' beside function symbols rather than through them (i.e. <T and T> rather than $\rlap{<}\mathrm{T}$ and $\rlap{>}\mathrm{T}$). Further discussion of *Funktionbezeichnungen* is deferred until Chap. 5, where we shall take them up in connection with works from Riemann's middle and late periods.

depth, but we shall begin to discuss it in this chapter since the idea was present in Riemann's earliest work.

3.4 Harmonic Function: Category or Chord

What is harmonic function? Riemann adumbrated a definition of sorts in 1893 when he spoke of harmonic "significance within the key" and asserted tonic, dominant, and subdominant, as "the only three kinds of tonal functions."[8] But he never probed the relation between chord and function as deeply as he should have, and left behind a theory whose tonal functions double unhappily as chords. This has troubled recent scholars, most notably Dahlhaus (1967), who faults Riemann for not distinguishing consistently between chord and function:

> The metaphor "main pillar" [*Hauptsäule*], of which it is uncertain whether it implies "function" or "chord," conceals an irresolvable difficulty in Riemann's formulation of the theory of functions. Riemann leaves undecided the question of whether "tonic," "dominant," and "subdominant" are terms for chordal scale degrees or for functions. The difference between appearance and significance, between what is presented and what is represented, is left up in the air.[9]

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[8] Riemann, *Harmony Simplified*, 9.

[9] Carl Dahlhaus, *Untersuchungen über die Entstehung der harmonischen Tonalität* (Kassel: Bärenreiter-Verlag, 1967), 42; trans. Robert O. Gjerdingen as *Studies on the Origin of Harmonic Tonality* (Princeton: Princeton Univ. Press, 1990), 50. [Die Metapher 'Hauptsäule', von der nicht festseht, ob sie 'Funktion' oder 'Akkord' bedeutet, verdeckt eine Schwierigkeit, die in Riemanns Formulierung der Funktionstheorie unaufhebbar ist. Riemann läßt es unentschieden, ob 'Tonika', 'Dominante', und 'Subdominante' Bezeichnungen für Akkordstufen oder für Funktionen sind. Die Differenz zwischen Erscheinung und Bedeutung, Präsentem und Repräsentiertem, wird in der Schwebe gehalten.]

The ambiguity that Dahlhaus describes was the result of Riemann's equating concrete chords (or scale degrees) with abstract categories such as tonic and dominant. Riemann's symbol T stands for "tonic function" but also for the major triad whose root is "I" of the prevailing key. Chords obviously express harmonic functions in the sense that C–E–G expresses T in C major, but chord and function are not thereby equivalent. C–E–G is no more tonic than A–C–E, E–G–B, or any other combination of pitches that might be heard in C major. Function points toward the behavior of chords—their "attitude" Hauptmann would say—not their material content. It follows that function symbols cannot be made to do the work of roman numerals or figured-bass notation without undermining their sense as function symbols. To the extent that Riemann wanted it both ways, his symbols were caught in a limbo between fundamental bass and Hauptmannian dialectic, between strict denotation and fairly liberal connotation. The chord-versus-category distinction is useful because it lets us group diverse chords functionally and—as context often demands—assign different functions to single chords. Riemann's theory accommodates a legion of chords but specifies just three harmonic functions. How was it possible for such apparently distinct concepts as chord and category to have coalesced in his mind? One answer lies in the historical relationship between *Stufe* (scale degree) theory and the theory of harmonic functions.

The now common practice of using roman numerals to denote chordal *Stufen* was the innovation of Vogler (1802),[10] but owed its initial and widest dissemination to Weber's popular *Versuch* (1817–21).[11] Hindsight shows us that *Stufe* theory—as advanced by Vogler and Weber, and refined by Sechter and Schenker—appealed to Austrian theorists, whereas function theory appealed to Germans.[12] This generalization reflects real differences between Austrian and German musicians in the nineteenth century, but obscures historical continuities that met in Riemann's work and accounted for his readiness to identify scale degree with function.

Riemann did not believe that roman-numeral analysis (which he credited to Weber) was essentially different from functional analysis. Both used abstract symbols to describe chords, and the differences between these methods were of degree rather than kind. Nearly a decade after the debut of the theory of harmonic functions Riemann wrote that his "designation of tonal functions was nothing more than a simplification, elaboration, and deepening of Weber's *Stufenziffern*."[13] It made sense for him to identify S, D and T with Weber's IV, V, and I, because these scale degrees

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[10] Abbé George Joseph Vogler, *Handbuch zur Harmonielehre und für den General baß, nach den Grundsätzen der Mannheimer Tonschule* (Prague: Karl Barth, 1802). Roman numerals appear in chap. 6, "Karaktere der Harmonien und Ausweichungen," 111–12.

[11] Gottfried Weber, *Versuch einer geordneten Theorie der Tonsetzkunst zum Selbstunterricht mit Anmerkungen für Gelehrtere*, 3 vols. (Mainz: B. Schott, 1817–21); 3d rev. ed., 4 vols. (Mainz: B. Schott's Söhne, 1830).

[12] See Robert Wason, *Viennese Harmonic Theory from Albrechtsberger to Schenker and Schoenberg* (Ann Arbor: UMI Research Press, 1985).

[13] Hugo Riemann, "Zur Reform der Harmonie-Lehrmethode," in *Präludien und Studien*, vol. 3 (Leipzig: Hermann Seeman, 1901), 47–68. The full reference reads (p. 64): "Wie meine Bezifferung überhaupt als eine Weiterbildung der Weberschen, mit prinzipieller Hineinarbeitung des dualen Prinzips (Zarlino, Tartini, Hauptmann, v. Oettingen) anzusehen ist, so ist auch die in meinem neuesten Buche (Der '*Vereinfachten Harmonielehre*') angewandte Bezeichnung der tonalen Funktionen nichts anderes als eine weitere Vereinfachung und doch zugleich Erweiterung und Vertiefung der Weberschen Stufenziffern." For the relation between Riemann and Weber, see Imig, 22–28.

conventionally expressed those functions. This is not to say that harmonic function was restricted to triads built on IV, V, and I. The *Leittonwechselklänge* and *Parallelklänge*, built variously on II, III, VI, and VII, were functional, as were more dissonant formations built on IV, V, and I. But all of these chords derived functionality from within a system that assumed IV, V, and I were archetypes and somehow interchangeable with S, D, and T.[14]

Riemann never explained his motive for grafting scale degree to function in this manner. Probably it was pedagogical. The appeal of a theory that was up-and-running (and selling textbooks), stood in contrast with earlier dead-end efforts to treat function categorically (see 3.5–3.7). However, Riemann's reduction of S, D, and T to a kind of scale-degree logic came at a price. Functional categories were well-suited to the analysis of nineteenth-century music, where conventional relations between scale degree and function were often dismantled. Roman-numeral analysis confronts this repertoire poorly, in part because diatonic *Stufen* can drift out of phase with perceived S, D, and T functions. Example 3-2 illustrates such phase-shifting over three versions of the Grail motive from *Parsifal.* Lewin (1984) uses this example to argue that "the nature and logic of Riemannian tonal space are not isomorphic with the nature and logic of scale-degree space."[15]

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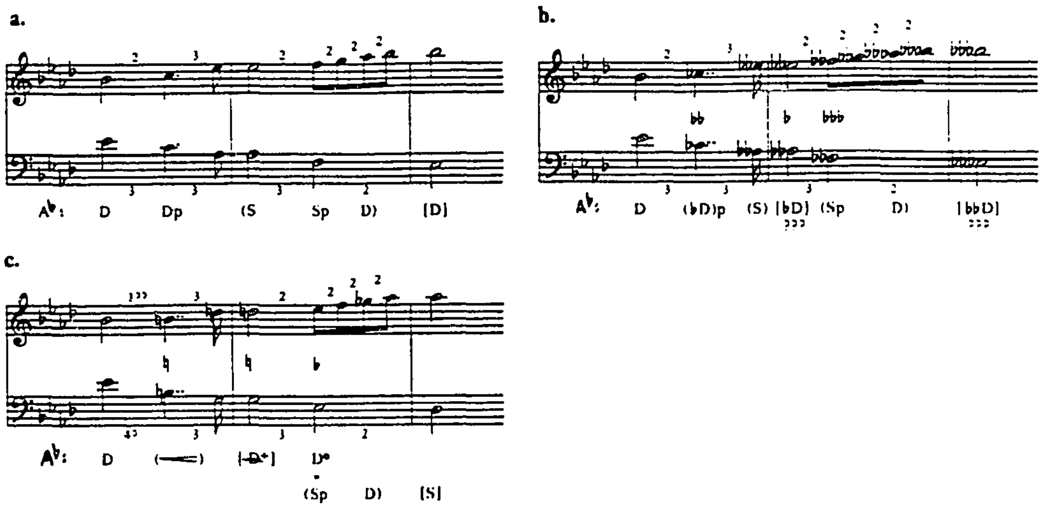
[14] See Dahlhaus, *Studies on the Origin of Harmonic Tonality* , 50. Dahlhaus makes this point when he says that "the theory of harmonic functions is a rigorous theory of fundamental progressions in which the number of degrees shrinks to just three (I, IV, V)."
[15] David Lewin, "Amfortas's Prayer to Titurel and the Role of D in *Parsifal*: The Tonal Spaces of the Drama and the Enharmonic C^{b}/B," *19th-Century Music* 7/3 (1984): 345.

EXAMPLE 3-2: LEWIN'S ANALYSIS OF THE GRAIL MOTIVE FROM *PARSIFAL*



A diatonic statement is given in 3-2a, followed by chromatic variants in 3-2b and 3-2c. By scale-degree space Lewin means the diatonic intervals between scale degrees in A^{b} major. Chromatic notes are alterations of diatonic ones with the same letter name: E^{b}, E^{bb}, E^{bbb}... are all versions of "V", likewise A^{b}–E^{b}, A^{b}–E^{bb}, and A^{b}–E^{bbb} are all "fifths." By Riemannian space Lewin means the functional intervals between chords. Lewin expresses the intervals of scale-degree space with numbers—2 and 3 for species of second and third—and those of Riemannian space with function symbols. The prototype Grail motive in 3-2a prolongs E^{b}, and the relation between scale degree and function is conventional: E^{b} expresses the function D, A^{b} expresses the function (S)D, and so on. Convention is skewed in 3-2b, which preserves the scale-degree space of 2a but not the Riemannian space. An attempt to carry over the functions of 3-2a demonstrates the out-of-phaseness: A^{bb} in m. 1 coincides with D not S; E^{bbb} in m. 3 coincides with S not D. In functional terms, 3-2a prolongs D and 3-2b moves from D to S. Example 3-2c captures the D-to-S-ness of 3-2b, but distorts scale-degree space in the process.

Lewin's point requires careful wording. The scale degrees of Examples 3-2a and 3-2b do not support different functions. Nor do the functions of 3-2b and 3-2c support different scale degrees. Scale degree and function coexist independently in these examples. Such coexistence may take different forms, but whether it is conventional (3-2a) or not (3-2b) the intervallic relations in each space are intrinsically different. Lewin says that the "objects and relations that Riemann isolates and discusses are *not* simply the old objects and relations dressed up in new packages; they are essentially different objects and relations, embedded in an essentially different geometry."[16] E^{bbb} in 3-2b does not express D; yet it does not express S either, since it does not exist in the Riemannian space of Example 3-2c. The corollary D^{b} in 2c expresses S but does not exist in the scale-degree space of 3-2b. Though 3-2b and 3-2c sound the same, each interprets the Grail motive from its own perspective: 3-2b bends Riemannian space to the scale-degree space of 3-2a; 3-2c bends scale-degree space to the D-to-S functionality of 3-2b. The effect of the passage lies in our awareness of both readings, and particularly in the realization that the scale-degree relations of 3-2a are somehow accompanied by new functions. The acoustic identity (not enharmonic equivalence) of 3-2b and 3-2c allows Wagner to realign the domains of scale degree and function, to "shift space" in effect.

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[16] Lewin, "Amfortas's Prayer," 345 (author's italics).

Lewin's "Riemannesque" analysis sheds light on tonal effects such as this, by separating scale degree and function. That Riemann did not appreciate or exploit such separation is the paradox at the heart of Lewin's observation that the "virtue and power of Riemann function theory, which is also the source of its problems and difficulties, is precisely its ability to avoid assigning letter names (i.e., implicit scale-degree functions) to its objects."[17] By conjoining scale degree and function Riemann converted an inherent strength of his theory into a perplexing weakness.

3.5 Musical Logic and "Musikalische Logik"

Our critique has thus far shortchanged Riemann's earliest work, which did attempt to treat function categorically. In the next several sections we shall examine the harmonic ideas set forth in "Musikalische Logik" and "Ueber das musikalischen Hören."

The word *Funktion* does not occur in the essay "Musikalische Logik," which was published pseudonymously one year before Riemann presented his dissertation at Göttingen. Nor does it occur in the dissertation "Ueber das musikalischen Hören," which introduced the ideas of *Tonvorstellung* and harmonic undertones. One misses the word in both places, since it is these works that uphold the idea of harmonic function in its purest sense. Instead of function, Riemann repeatedly invoked the phrase "musical logic." The word "logic" has a syntactical sense that brings Oettingen to mind, but musical logic was in fact a reference to Hauptmann's metaphysical concepts of *Octaveinheit* (octave

[17] Lewin, "Amfortas's Prayer," 344.

unity), *Quintentzweiung* (fifth division), and *Terzeinigung* (third unification).[18] Riemann interpreted these concepts functionally and renamed them thesis, antithesis, and synthesis. By applying these terms to cadential archetypes, he determined the functional significance of chords—whether they were thetic, antithetic, or synthetic—and their logical succession in musical structures. Musical logic was a dialectical logic.

Riemann believed however that musical logic was rooted in laws of human psychology and perception. Function was not merely taxonomic, it was interpretive and tied to innate mental processes. Musical logic (and music by extension) was a kind of sounding template of the mind. Riemann's technical knowledge of perception came almost entirely from Helmholtz, whose *Lehre von den Tonempfindungen* had singlehandedly established the field of psychoacoustics and placed music theory on unprecedented scientific footing. In the first part of this great work, Helmholtz expounded the physiological process of perception. These ideas were the springboard for Riemann's own views, which diverged considerably from Helmholtz's.[19] One difference apropos of musical logic was Riemann's emphasis on the active-comparative nature of perception as against Helmholtz's essentially passive view. Riemann

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[18] Hauptmann (pp. 21–32) used a variety of terms for the three stages of his dialectic: *Quintbegriff* and *Quintzweiheit* are altervatives to *Quintentzweiung*, and all three express *Zweiheit* (duality) or *Trennung* (separation); *Terzbegriff* and *Terzeinigung* likewise express *Einheit der Zweiheit* (union of duality) or *Verbindung* (union). The terms thesis, antithesis, and synthesis, though commonly associated with Hauptmann, do not occur in his text.

[19] Hermann von Helmholtz, *Die Lehre von den Tonempfindungen als physiologische Grundlage für die Theorie der Musik.* Braunschweig: Fr. Vieweg & Sohn, 1863; trans. A. J. Ellis from the 4th German ed. (1877) as *On the Sensations of Tone as a Physiological Basis for the Theory of Music* (London: Longmans, 1875; reprint, New York: Dover, 1954). Sections 4.2 and 4.3 of the present study outline Helmholtz's theory of music perception.

believed that the function of a musical sound was determined by its relation to other sounds. Because functional judgements were contextual, they were subject to change. At some point one might compare groups of sounds—phrases with phrases, periods with periods—instead of individual sounds. Riemann's point was that perception does not simply register detached sounds, but responds intelligently and flexibly to musical context. Paramount to this conception was, clearly, memory. So was the idea that listeners were predisposed to hear tonally. Indeed, tonality itself arose through a mental comparison of sounds, and the attendant desire to assign primary significance to one of them.[20]

The musical logic of Riemann's dissertation years was excellently suited to his comparative view of music perception: Thesis, antithesis, and synthesis had no connotation of scale degree and could be applied to groups of chords as easily as individual chords. They provided a suggestive and versatile means for expressing general intuitions of music's flow through time. Riemann's attention to both the temporal aspect of music and higher-level intuitions was characteristic of his work. Reflecting on his career in 1914, he wrote that the localized "bottom-to-top [or] inductive method" ("von unten nach oben, die 'induktiv Methode'") of Helmholtz and Stumpf had mired them in "the detail work of preliminary tone-psychological investigations." Riemann himself had opted for a more general "top-to-bottom, [or] deductive method." ("von oben nach unten, die 'deduktive Methode'").[21] His frustration with traditional music theory was its failure to attend to general musical-logical

[20] See Riemann, "Ueber Tonalität," 443–44, 453–54; and Mark McCune, "Hugo Riemann's 'Ueber Tonalität': A Translation," *Theoria* 1 (1985): 132–50.
[21] Riemann, "Ideen zu einer 'Lehre von den Tonvorstellung,'" *Jahrbuch der Musikbibliothek Peters* 21–22 (1914/15; reprint, Wiesbaden: Lessing, 1965), 1–2.

principles that explained the temporal succession (*zeitlichen Nacheinander*) of musical events and the structural impression left by those events. The frustration with "detail work" at the expense of general principles was evident in the opening sentence of "Musikalische Logik," where Riemann justified his "gänzlich neuen Gesichtpunkt" on the basis of errors he had often perceived in music—errors of which traditional "harmony and counterpoint [could] provide no evidence."[22] It will help to draw attention to some points in "Musikalische Logik" that do not come through as clearly in the dissertation. The essay appeared in two installments that were titled "Harmonische Logik" and "Metrische Logik," respectively.[23] Our comments refer only to "Harmonische Logik," where Riemann treated function under the following headings: 1) *Cadenz* (cadence); 2) *Nebenharmonien* (secondary harmonies); 3) *Erweiterte Cadenzen* (expanded cadences); 4) *Modulation* (modulation); and 5) *Quintenverbot* (forbidden fifths). Besides the term "function," there were no references to harmonic dualism or to the undertone series in this essay. Nor did the names of Helmholtz or Oettingen appear. Riemann surely knew the work of these scientists, but "Musikalische Logik" attests the singular influence of Hauptmann.

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[22] Riemann, "Musikalische Logik," 279. [Die Thatsache, daß ich oft beim Anhören eines Musikstückes Fehler empfand, wo Contrapunkt und Harmonielehre keine nachweisen konnten, wurde der Anstoß zu den nachfolgenden Untersuchungen, die ein bisher fast gänzlich unbebautes Feld urbar zu machen suchen.]

[23] On the relation between meter and harmonic function, see William Caplin, "Tonal Function and Metrical Accent," *Music Theory Spectrum* 5 (1983): 1–14.

3.6 *Große Cadenz* as Paradigm

Under *Cadenz* Riemann gave a functional analysis of the harmonic progression shown in Example 3-3.[24] Referred to as the *große Cadenz,* this five-chord structure was considered by Riemann to be "der Typus aller musikalischen Form."[25] Riemann hoped that by interpreting it functionally, he would then be able to show how the same functions underlay more complex progressions. He acknowledged the permissiveness of current practice—Oettingen's "chaos of possibilities" was apparently the state-of-the-art—but believed that musical-logical meaning imposed a "limit" (*Schranke*) in all music, since the three harmonic functions were really just characteristic modes of perception. Despite the renegade impression of modern harmony, there was a deeper order that Riemann aimed to make explicit. He promised that the "freest and most complicated harmonic... formations will be seen to derive from the simplest principle of thesis, antithesis, and synthesis."[26]

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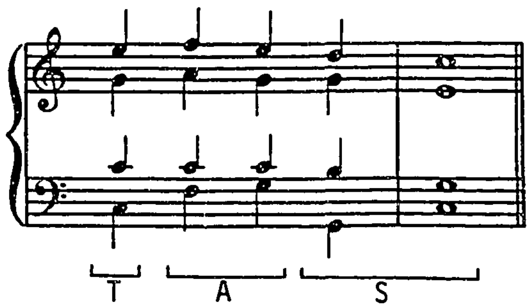
[24] Riemann's grouping of the cadential six-four with the subdominant is at odds with the usual reading of this chord as an embellishment of dominant harmony.
[25] Riemann, "Musikalische Logik," 280.
[26] Riemann, "Musikalische Logik," 280.

EXAMPLE 3-3: THE *GROßE CADENZ*



The "simplest principle of thesis, antithesis, and synthesis" was embodied by the *große Cadenz*. Riemann's analysis of the progression assumed general conditions of stability, which he asserted but did not define: The most stable chord was the initial tonic, which Riemann analysed as thesis (Hauptmann's *Octav-* or *Einheitsbegriff*); the subdominant and tonic six-four challenged this stability and were thus analysed as antithesis (Hauptmann's *Quintbegriff*); the dominant and final tonic reinstated harmonic stability and were analysed as synthesis (Hauptmann's *Terzeinigung*).

Though Riemann and Hauptmann clothed their descriptions of this cadence in similar language, Riemann's understanding was quite different from Hauptmann's. For Hauptmann, the cadence embodied a conception of key that was neither perceptual nor even explicitly temporal. The order of chords was secondary to the fact that "die Tonart von allen Seiten gezeigt ist."[27] One could say that Hauptmann's conception was primarily spatial.[28] Riemann was not as aloof from his material, and had no interest in consigning chords to a sort of metaphysical stasis. The cadence (and

[27] Riemann, "Musikalische Logik," 280.
[28] See 2.3 for Hauptmann's conception of major and minor harmony.

tonality itself) was emphatically temporal: Logical meaning was the meaning that arose when chords were sounded and heard one after another.

The most noteworthy feature of Riemann's analysis was his assignment of individual functions to pairs of chords. Though the *große Cadenz* contained five chords, Riemann isolated just three functionally "hervortretende Momente." One result of this was that the tonic chord was assigned a new function for each of its three appearances. Riemann spoke the language of Hauptmann here, claiming to see the "*Quintbegriff* in the second appearance of the tonic, which is set against the *Einheitsbegriff* of the first appearance, and which finds its *Terzeinigung* once again in a root-position tonic."[29] Interestingly, he altered this passage in the dissertation, replacing Hauptmann's terms with his own *These, Antithese* and *Synthese*, and noting parenthetically that his analysis is not restricted to *Klangempfindung* but treats the connection (*Verknüpfung*) of individual *Vorstellungen* by means of "ein geistiges Band." *Klangempfindungen* refer to the harmonic sensations addressed in Helmholtz's *Lehre von den Tonempfindungen*, whereas *Vorstellungen* are Riemann's more sophisticated music-psychological representations; *Verknüpfung* connotes a logical or systematic connection, and occurs throughout Riemann's work in discussions of chord progression. Both the original and altered versions of Riemann's text follow:

[29] Riemann, "Musikalische Logik," 280.

> I see in this second appearance of the tonic the *Quintbegriff* [fn. these expressions are chosen in connection with Moritz Hauptmann], which opposes the *Einheitsbegriff* of the first appearance and—by means of the *Oberdominant*—finds its *Terzeinigung* again in the root position tonic. This formation is the archetype of all musical form. ("Musikalische Logik," 280.)[30]

> I see in this appearance of the tonic following the *Unterdominant*, the *Antithese* (in the Hauptmannian sense, manifest here according to my own view that the point of theory is no longer the mere analysis of *Klangempfindung*, but rather of the mental linkage of individual *Vorstellungen*); this opposes the *These* of the first appearance and—by means of the *Oberdominant*—finds its *Synthese* again in the root position tonic. This formation is the archetype of all musical form. ("Ueber das musikalische Hören," 52.)[31]

Though he was thinking function-as-category when he analysed the *große Cadenz*, Riemann nonetheless confounded category with chord. This was an inevitable consequence of presuming a structural archetype to begin with. On one hand, function "interpreted" the cadence: Categories were distinct from chords and Riemann could call any chord (or group of

[30] Riemann, "Musikalische Logik," 280. [Ich sehe in diesem zweiten Auftreten der Tonika den Quintbegriff (Diese Ausdrücke sind in Anschluß an Moritz Hauptmann gewählt) der sich dem Einheitsbegriff des ersten Auftretens entgegensetzt, und der seine Terzeinigung durch die Oberdominant wieder in der Tonika findet, die nun wieder in der Grundlage erscheint. Diese Gestalt ist der Typus aller musikalischen Form.]

[31] Riemann, "Ueber das musikalische Hören," 52. [Ich sehe in diesem Auftreten der Tonika nach der Unterdominante die Antithese (in Hauptmann'scher Auffassungsweise, die meiner Ansicht nach, wo nicht mehr nur von blosser Analyse einer Klangempfindung, sondern von Verknüpfung einzelner Vorstellungen durch ein geistiges Band die Rede ist, durchaus am Platze erscheint), die sich der These des ersten Auftretens entgegensetzt, und die ihre Synthese durch die Oberdominante wieder in der Tonika findet. Diese Gestalt ist der Typus aller musikalischen Form.]

chords) by one of three names. On the other hand, the cadence "interpreted" harmonic function by imposing an archetypical syntax on supposedly neutral chord categories. It was hard to keep chord and function apart under these circumstances and Riemann occasionally faltered. After identifying the three harmonic functions and interpreting the three statements of tonic harmony, Riemann gave this summary:

> Thesis is the first tonic, antithesis the *Unterdominant* along with the tonic six-four, and synthesis the *Oberdominant* along with the root position tonic; the tonic is thetic, the *Unterdominant* is antithetic, and the *Oberdominant* is synthetic.[32]

Chord-pairs were included in the definition of function ("Unterdominante mit dem...") and then excluded from it ("antithetisch die Unter-..."). In the course of a single statement, Riemann interpreted function both as category and chord.

Chord and function were drawn together at other points in "Musikalische Logik." Riemann claimed that the tonic six-four brought

32 Riemann, "Musikalische Logik," 280. [These ist die erste Tonika, Antithese die Unterdominante mit dem Quartsextaccord der Tonika, Synthese die Oberdominante mit dem Grundaccord der Tonika; thetisch ist die Tonika, antithetisch die Unter-, synthetisch die Ober-Dominante.] The paragraph begins rather obscurely: "This so-called große Cadenz thus has three especially prominent moments, which we shall characterize thetically: Thesis, thetic fifth, and thetic third." [Diese sogenannte große Cadenz hat also drei besonders hervortretende Momente, die wir thetische nennen wollen: These, thetische Quint und thetische Terz."] Harrison (p. 267 n. 28) suggests that the entire cadence may be thetic at a higher level, in which case the local antithetic and synthetic moments are reinterpreted as thetic fifth and thetic third. This is probably granting Riemann too much. The passage in question was emended in the dissertation to read: "Diese sogenante [*sic*] grosse Cadenz hat also drei besonders hervortretende Momente, die wir thetische nennen wollen: These, Antithese und Synthese."

"nothing new" to the cadence, and that "IV [was] the actual antithesis to I."[33] By the time *Nebenharmonien* were introduced (chords on II, III, VI, and VII), a privileged relation already existed between I, IV, and V and the three functional categories. The logical meaning of *Nebenharmonien* depended on their relationship with one or another of the *Hauptaccorde*, which Riemann demonstrated by clustering II, III, VI, and VII around I, IV, and V in various contexts. The segregation of key into *Hauptaccorde* and *Nebenaccorde* was incompatible with the function-as-category concept, but was unavoidable as long as the *große Cadenz* was maintained as a paradigm.

Despite such problems, there remains a lingering sense of categorical function in this essay. Riemann's language suggests the distinction, however fleetingly, between synthesis and V, antithesis and IV, thesis and I. The thetic terminology is too vague to describe chordal function and in later works Riemann's terminology is the first thing to change when chord and function are merged. Traditional terms such as *Tonic*, *Unterdominant*, and *Oberdominant*, which close the gap between scale degree and function, replace the vague "attitudinal" language of earlier work. What is left of categorical function in "Musikalische Logik" is found in the section on *Erweiterte Cadenzen*, a section whose omission from the dissertation was a clear signal of things to come.

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33 Riemann, "Musikalische Logik," 280.

3.7 Expanded Cadences

Once Riemann had introduced the *große Cadenz* and explained the various meanings of *Nebenharmonien,* he was ready to consider more elaborate structures. Four modes of *Erweiterung* were responsible for the set of expanded cadences, which at the highest level included full-length compositions. We shall call these modes 1) extension; 2) repetition; 3) cadence-within-cadence; and 4) mixture.[34] *Erweiterung* by extension expanded the cadence by repeating a primary chord, or inserting a chord of "gleicher logischer Bedeutung" into one or more of the functional moments. *Erweiterung* by repetition expanded the cadence by repeating the antithetic or synthetic moment. Cadence-within-a-cadence expanded one moment—usually the thetic moment—into a subordinate cadence within the *große Cadenz.* Mixture was some combination of 1, 2, and 3.

Riemann's own examples of *Erweiterung* are reproduced in Example 3-4. Notice that he used roman numerals to represent scale degree and a bracket notation to represent function. Though he did not label the brackets, their functional meaning was clear from his discussion of *Nebenharmonien*: I–VI in the first progression is thesis, IV–VI is antithesis, and III–V is synthesis. The examples of repetition (2) show two chord pairs (II–VI, IV–I) functioning antithetically, and two others (V–I, VII–I) functioning synthetically. A thetic cadence-within-a-cadence follows in the third progression, and the fourth progression is a mixture of all three types of *Erweiterung.* Riemann's use of different notations to

[34] Riemann's terms are "Verweilen," "Wiederholung," "Cadenz innerhalb der Cadenz," and "Mischung." Harrison (p. 269) calls cadence-within-a-cadence "embedding" and does not mention *Erweiterungen* that result through mixture.

represent function and scale degree suggests that the concepts were still distinct for him. This section of "Musikalische Logik" was excised from the dissertation, and bracket notation never reappeared in subsequent works.

EXAMPLE 3-4: RIEMANN'S *ERWEITERUNGEN*

1) durch längeres Verweilen auf einem Momente, indem derselbe Accord in verschiedenen Lagen erscheint, oder statt dessen andere von gleicher logischer Bedeutung eingeschoben werden, z. B.

I, VI – IV, VI – III-V – I –

2) durch Wiederholung der Antithese und Synthese:

I–II – VI–IV–I–V – I

oder I–IV–I–V – I–VII–I

3) durch Erweiterung eines Momentes, besonders des thetischen zur Cadenz innerhalb der Cadenz:

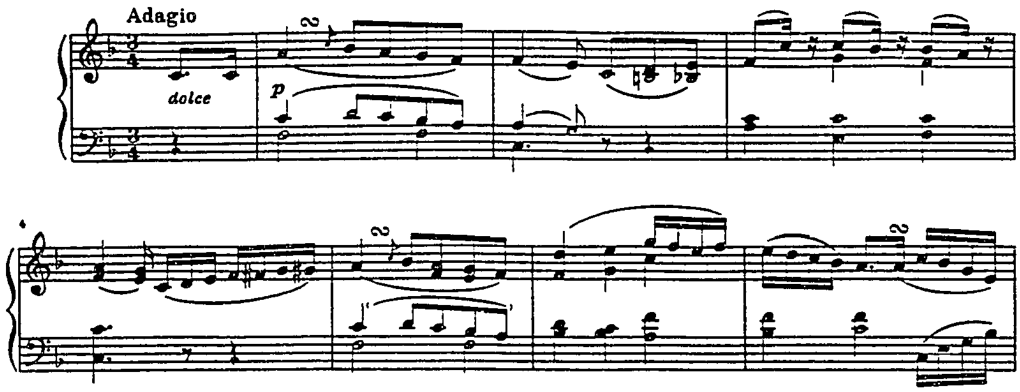
I – V – I –IV–I – V–I

4) durch Mischung aller drei Arten:

I–V–I– IV–I–II–VI, I –V, VII–I

Riemann described a special idiom of *Erweiterung* in connection with antecedent-consequent structures. He said that composers frequently expanded the thetic moment of a musical period (through extension or cadence-within-cadence) and then concluded with an *einfache Cadenz* I–IV–V–I. The opening measures of the second movement of Beethoven's Piano Sonata, op. 2, no. 1 provide an example of such thetic expansion. Riemann gave the "sehr complicirte Schema" reproduced below Beethoven's music in Example 3-5, and the following analysis: "Until measure 5, the first phrase is nothing but an expanded thesis; measures 5–6 provide a complete cadence with stress on IV, as do meaures 6–8, which conclude finally with a suspension on the tonic."[35]

EXAMPLE 3-5: FUNCTIONAL ANALYSIS OF OP. 2, NO. 1



ersten acht Takte von Beethoven's Adagio der ersten Clavier-sonate folgendes sehr complicirte Schema entwerfen: Harmonisirung: I—V—I, V, I — V — || — I—IV, V, I — IV, I, V — V, I. Das erste Sätzchen bis Takt 5 ist nichts

Riemann did not restrict his analysis of op. 2, no. 1 to chords, but went on to show how musical logic was played out in the melody. Melodic tones were interpreted as either "Grundton, Terz...sogar als Quint eines Accordes."[36] This idea of single tones implying complete chords was later

[35] Riemann, "Musikalische Logik," 282. [Das erste Sätzchen bis Takt 5 ist nichts als erweiterte These, 5–6 vollständige Cadenz mit dem Ton auf IV, 6–8 noch einmal dieselbe, wieder mit betonter Unterdominant und schließlich mit einem metrischen Vorhalte vor der Tonika.] Riemann's intentions are a little unclear; "bis Takt 5" apparently means up to and including measure 5, which would seem to function as both the ending of a thetic expansion and beginning of an antithetic one (the two cadences in mm. 5–8, which stress IV).
[36] Riemann, "Musikalische Logik," 281–82.

called *Klangvertretung*. Though Riemann did not use the term here, or consider the psychological implications of the idea, by the end of the decade *Klangvertretung* would become an important supplementary rationale for minor harmony. Example 3-6 gives Riemann's melodic analysis of mm. 1, 2, and 4 of the Beethoven excerpt. The roman numerals represent implied chordal roots for the melodic tones of the antecedent phrase (refer to Example 3-5). The pitch A4 in the melody implies F major (roman numeral I), B^{b}4 implies B^{b} major (roman numeral IV), A4 implies F major (roman numeral I), G4 implies E diminished (roman numeral VII), and so on.[37] The brackets show that the melody of m. 1 expresses thesis, whereas that of m. 2 expresses antithesis and synthesis.[38] In m. 4 Riemann interprets A4, G4, and C4 as an expanded thesis, the passing tone D4 as antithesis, and the chromatic ascent from E4 to G$^{\#}$4 as synthesis.

EXAMPLE 3-6: MELODIC ANALYSIS OF OP. 2, NO. 1

Takt 1: I, IV, I, VII, I, || IV, I, V, I
These. A. S.
Takt 4: I, V, I, IV, V, I, I♯. VII. VII♯ I. Nach dem
T. A. Synthese.

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[37] Here and elsewhere in this chapter, specific pitch is indicated with the notation adopted by the Acoustical Society of America: Cello C is C2, viola C is C2, middle C is C4, and so on.

[38] Riemann's analysis of m. 2 does not line up with the melody; there are five melodic pitches in this measure, to which Riemann assigns only four roman numerals: IV, I, V, I,

We shall not bother with the shortcomings of this (Riemann's first published) analysis. The real point of interest is Riemann's hierarchical conception of function. At the lowest level he describes melodic function, he then interprets the harmonies of each measure, and finally describes a thetic moment that lasts for five measures.[39] This five-measure thesis extends through the half cadence in m. 4, and thereby emphasizes the motion between phrases instead of the articulation (which Riemann indicates with two vertical lines).[40]

3.8 Musical Logic and Linguistics

In the previous section we resisted the term "generative" in connection with the different modes of *Erweiterung*. Let us now consider Riemann's musical logic as a generative theory, or at least the blueprint for such a theory, in the sense of modern linguistics. We shall compare the incipient chord grammar of "Musikalische Logik" with the generative-transformational grammar advanced by Noam Chomsky in the late 1950s. Specifically we shall formulate the concept of *Erweiterung* along the lines of "rewrite" rules, a formalism that Chomsky developed in *Syntactic Structures* (1957) to describe phrase structure.

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[39] Harrison, 271 (figs. 6.1, 6.2) sketches various levels of embedding in this passage.

[40] Schoenberg's discussion of this music emphasizes phrase articulation. See *Fundamentals of Musical Composition* (London: Faber and Faber, 1983), 25–26. Discrepancies between formal articulation and tonal motion have been treated extensively in the literature; see Fred Lerdahl and Ray Jackendoff, *A Generative Theory of Tonal Music* (Cambridge, MA.: MIT Press, 1983); William Rothstein, *Phrase Rhythm in Tonal Music* (New York: Schirmer Books, 1989); Peter H. Smith, "Brahms and Schenker: A Mutual Response to Sonata Form," *Music Theory Spectrum* 16/1 (1994): 78.

The goal of a grammar is to "generate all the grammatical sentences of [a language] and none of the ungrammatical ones."[41] Chomsky's criterion for grammaticalness was whether the sentences thus generated were "acceptable to a native speaker."[42] He believed that relatively few rules governed the infinite set of acceptable sentences in a language. The point of generative grammar was not literally to generate these sentences, but to describe in formal terms the capacity of anyone who knew the language to understand and speak them. By treating language behaviorally, Chomsky sought to make his enterprise a branch of psychology.

Riemann considered musical logic to be a branch of psychology, but his was also a linguistic enterprise. He spoke explicitly of the grammatical aspect of his work and suggested it more generally with such titles as *Musikalische Syntaxis* and "Musikalische Grammatik."[43] It is worth emphasizing that "Ueber das musikalische Hören" was republished as *Musikalische Logik* (see n. 2), a change in name that intimated the grammatical rather than perceptual aspect of music.[44] By contrast, Hauptmann, Helmholtz, and Oettingen all failed to work out a theory of

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[41] Noam Chomsky, *Syntactic Structures* (The Hague: Mouton, 1957), 13.
[42] Chomsky, 13.
[43] Riemann's interest in poetry doubtlessly contributed to the linguistic bent of his music theory. He spent much of his youth writing poetry, recited many of his poems publicly and received the approbation of such literary scholars as Wilhelm Scherer. An edition of poetry was prepared for publication in 1870 but interrupted by the Franco-Prussian war, after which Riemann took leave of poetry with the following verse: "Seitdem Musik sich meinem Geist erschloß/Hat sich der Poesie mein Herz verschlossen/Weil sich in süßer Töne Strom ergoß/Was sonst in guter Rede Fluß geflossen./Doch ist es heute wie es war geblieben:/Ich bin ein Mensch und bin der Musen Sohn/Und wie ich vordem Liebe nur geschrieben/So klinget nun der Lieb ein jeder Ton." [Ever since music opened itself to my spirit/My heart has closed itself to poetry/A current pours forth in sweet tones/That river which used to flow in fine words/Yet today, as always:/I am a mensch, the muses' son/And as once I wrote of pleasing things/The pleasant sounds now in every tone.]
[44] The subtitle of this work *Hauptzüge der physiologischen und psychologischen Begründung unseres Musiksystems* seems a calculated reference to that of Helmholtz's work: (*Die Lehre von den Tonempfindung*) *als eine physiologische Basis zur Musiktheorie.*

chord progression that distinguished "grammatical" from "ungrammatical" in common practice harmony. Hauptmann was scarcely interested in the problem, and his inability to bring "drei Accorde in vernünftiger Reihenfolge nach einander"—a reasonable order of succession—was a legitimate source of dismay for Riemann. Chords were at least empirical for Helmholtz but his analyses proceeded almost entirely in isolation from musical context. Oettingen did present a theory of chord connection but this, as we have seen, lacked functional constraints and harbored unrealistic convictions about the phonic system.

Though "Musikalische Logik" did not present a formal theory, the three functions and four *Erweiterungen* constitute a grammar that satisfies Chomsky's desideratum surprisingly well. Syntactic analysis (also called constituent analysis) is generally formulated in terms of parsing rules, and parsing rules are in some sense essential to Riemann's description of phrase structure in music. The form of grammar assumed by such descriptions, whether in music or language, can be thought of as a set containing two elements that are themselves sets. Chomsky represents such a grammar with the notation $[\Sigma, F]$, where Σ is a finite set of "initial strings"—strings that symbolize some initial, unparsed object(s) of the language—and F is a finite set of "instructional formulae" that generates a description of Σ. The elements of F are generally of the form $X \rightarrow Y$, which Chomsky interprets as "rewrite X as Y," and which we shall here call "rewrite rules." Example 3-7 adapts a simple phrase structure grammar from *Syntactic Structures*, where Σ contains the element *Sentence*, and *F* contains rewrite rules (i) – (vi).[45]

EXAMPLE 3-7: PHRASE-STRUCTURE GRAMMAR

Rewrite rules: $X \rightarrow Y$ (rewrite X as Y)

(i) $Sentence \rightarrow NP + VP$
(ii) $NP \rightarrow T + N$
(iii) $VP \rightarrow Verb + NP$
(iv) $T \rightarrow$ *the*
(v) $N \rightarrow$ *man*, *ball*, etc.
(vi) $Verb \rightarrow$ *hit*, *took*, etc.

By applying *F* {(i), (ii)... (vi)}, we derive the sentence "the man hit the ball" in Example 3-8a. Beginning with Σ {*Sentence*} in other words, we form successive strings so that each new one is derived from the previous one by applying some rule of *F*: The second string is formed by (i), the third string by (ii), and so on. The parsed *Sentence* of Example 3-8a is in this sense a derivation of [Σ, *F*]. Since the final string "*the* + *man* + *hit* + *the* + *ball*" cannot be rewritten further in [Σ, *F*], it is called a terminal string. Example 3-8a gives just one of the possible terminal derivations for [Σ, *F*], where Σ contains the element {*Sentence*} and *F* contains six rewrite rules. Chomsky is only interested in grammars of the form [Σ, *F*] that yield terminal strings: grammars that describe languages in the sense that Example 3-7 describes Example 3-8a.[46] The tree diagram of Example 3-8b simplifies 3-8a and enables a visual inspection of constituent strings—"hit

[45] Chomsky, 26.
[46] Chomsky, 30.

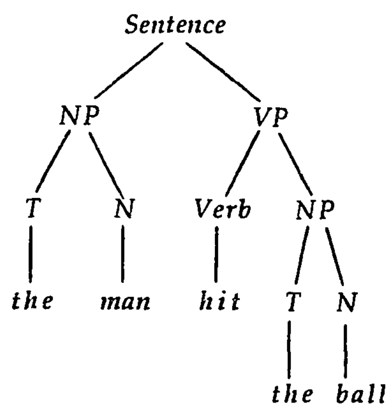
the ball" is constituent because all its words trace to a common node in the tree (*VP*); "the man hit" traces to diverse nodes (*NP* and *VP*) and is therefore nonconstituent. Notice that Example 3-8b gives no information about order of derivation. Any of the derivations of *F*{(i), (ii)... (vi)} will reduce to the same tree, which is to say that tree-sharing derivations are syntactically equivalent.

EXAMPLE 3-8: TERMINAL DERIVATION AND TREE DIAGRAM

A:

Strings	Rewrite rule
Sentence (initial string)	
NP+*VP*	(i)
T+*N*+*VP*	(ii)
T+*N*+*Verb*+*NP*	(iii)
the+*N*+*Verb*+*NP*	(iv)
the+*man*+*Verb*+*NP*	(v)
the+*man*+*hit*+*NP*	(vi)
the+*man*+*hit*+*T*+*N*	(ii)
the+*man*+*hit*+*the*+*N*	(iv)
the+*man*+*hit*+*the*+*ball*	(v)

B: Tree Diagram



Example 3-9a interprets the function theory of "Musikalische Logik" as a phrase structure grammar of the form [Σ, *F*], taking *große Cadenz* (*GC*) as the initial string and defining thirteen rewrite rules. This grammar is more complex than the one in Example 3-7, for we may derive different terminal strings depending on which (if any) *Erweiterungen* are involved. To indicate that *Erweiterung* may or may not occur we use parenthenses, so that *T*(*e*), *A*(*e*), and *S*(*e*) are read as "thesis with optional elaboration," "antithesis with optional elaboration," and "synthesis with optional elaboration." To indicate choice within each mode of *Erweiterung*, we use brace notation.[47] Braces enclose roman-numeral strings that are functionally equivalent, where roman numerals are understood to represent diatonic roots. Example 3-9b gives the terminal derivation for the prototype *große Cadenz* I–IV–I–V–I. Notice that this cadence is an optional equivalent to the *einfache Cadenz* I–IV–V–I. The incongruity of *große* and *einfache* structures occupying the same level in Riemann's syntax is due to the category-versus-chord dilemma: *große Cadenz* as paradigm, yet tonic six-four bringing "nothing new." Example 3-9c summarizes the rewrite rules of Example 3-9a, and Example 3-9d shows the derivation of the *große Cadenz* described in Example 3-9b. We see that I–VI/I–V–I is a constituent of *GC* because both progressions can be traced to a single node ($T(e)$) in Example 3-9c. The progression IV–I is also constituent (traceable to $A\,e^1$), but I–IV is not (I traces to T traces to $T(e)$; IV traces to A traces to A(e)).

47 Chomsky, 29 n. 3; 110. Chomsky introduces parentheses and brace notation to show linguistic options in more complex grammars.

EXAMPLE 3-9: PHRASE STRUCTURE GRAMMAR FOR "MUSIKALISCHE LOGIK"

A: Rewrite rules: $X \rightarrow Y$ (rewrite X as Y)

(i) $GC \rightarrow T(e) + A(e) + S(e)$

(ii) $T(e) \rightarrow \{T, Te^1, Te^2\}$

(iii) $T \rightarrow$ I

(iv) $Te^1 \rightarrow$ I–VI

(v) $Te^2 \rightarrow$ I–V–I

(vi) $A(e) \rightarrow \{A, Ae^1, Ae^2\}$

(vii) $A \rightarrow$ IV

(viii) $Ae^1 \rightarrow$ {IV–I, IV–VI}

(ix) $Ae^2 \rightarrow$ {II–VI–IV–I, IV–I–II–VI–I}

(x) $S(e) \rightarrow \{S, Se^1, Se^2\}$

(xi) $S \rightarrow$ V–I

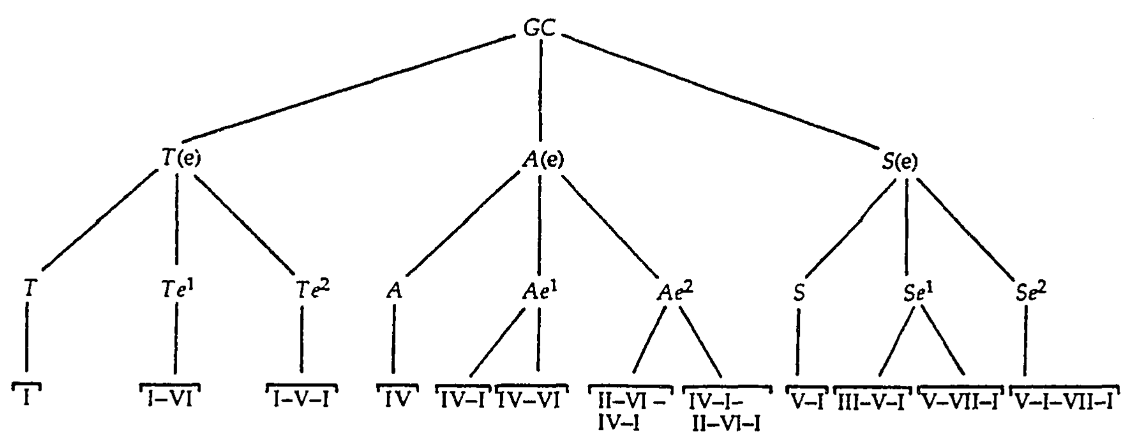
(xii) $Se^1 \rightarrow$ {III–V–I, V–VII–I}

(xiii) $Se^2 \rightarrow$ V–I–VII–I

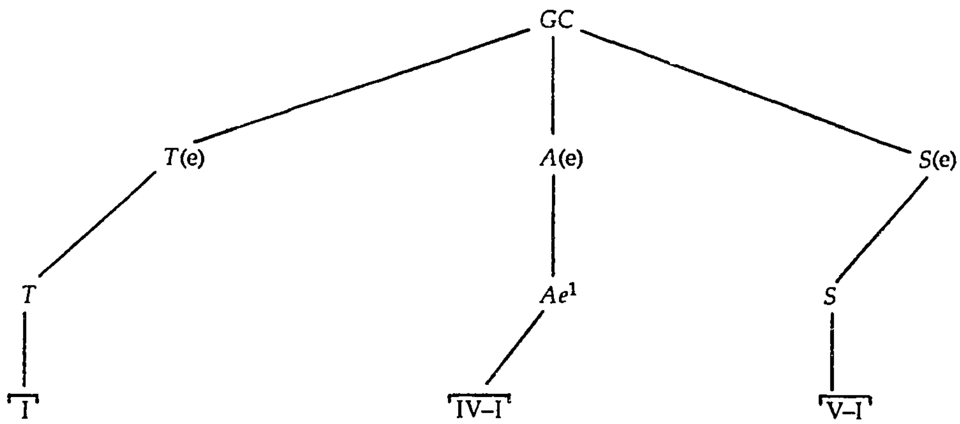
B:

Terminal Derivation	Rewrite Rule
GC (initial string)	
$T(e) + A(e) + S(e)$	(i)
$T + A(e) + S(e)$	(ii)
I + $A(e) + S(e)$	(iii)
I + $Ae^1 + S(e)$	(vi)
I + IV–I + $S(e)$	(viii)
I + IV–I + S	(x)
I + IV–I + V–I	(xi)

C: Tree Diagram of Rewrite Rules



D: Tree Diagram of *große Cadenz*



Example 3-9 is meant to be illustrative of Riemann's *Erweiterungen* (Ex. 3-4), however the scope of F {(i), (ii)... (xiii)} permits several terminal strings for which Riemann gives no account: for example, I–VI/IV–I–II–VI–I/V–I–VII–I ($T + A e^2 + S e^2$). On the other hand, the grammar is not yet general enough to describe the thetic I–V–I–V–I at the beginning of Beethoven's op. 2, no. 1, since there is no rule to allow for V–I suffixes to $T e^2$. Nor does the grammar describe formations that would result from concatenations of *Erweiterung* classes: $T e^1 + T e^2$, $A e^1 + A e^2$, $S e^1 + S e^2$. Finally, the grammar is "flat" hierarchically since the set Σ is limited to one initial string (the *große Cadenz* GC). A more robust grammar would include strings for higher and lower levels of structure: ..GC_{-2}, GC_{-1}, GC, GC^1, GC^2... where GC is a four-measure phrase, GC_{-1} a subphrase, GC^1 an antecedent-consequent, and so on. Such generalization is within the range of phrase structure grammar, but at a high cost of maintenance. As the linguistic scope becomes more inclusive, the number of elements in F increases until the grammar finally is overburdened with rules. Chomsky mitigates this problem by introducing a class of transformational rules; rules that handle common structural changes that would be laboriously defined in *F*. A robust musical grammar would include transformational rules to describe certain kinds of repetition, expansion, and elision, but this is far beyond our present concerns. To the extent we have shown that phrase structure grammar is an adequate model for function theory in "Musikalische Logik," we have shown the limitations that inhere in this harmonic conception.

Grammar was not an end in itself for Riemann or Chomsky, but a window onto human nature. Riemann treated the *große Cadenz* as the progenitor of musical logic. It embodied what he believed musicians knew intuitively about (tonal) music; the goal of music theory was to describe this knowledge as systematically as possible. Musical logic was hardly an algorithm for composition, but rather a theory for assigning coherent functional analyses to musical sentences. Chomsky's generative grammar similarly assigned parsing structure to verbal sentences. In both cases, theory gave form to obscure processes of human perception and cognition.

3.9 "Ueber das Musikalische Hören"

As a treatise on music perception Riemann's dissertation delivered less than its title promised. On the other hand, Riemann did christen the implicit psychology of "Musikalische Logik." The name he chose as a touchstone for his music psychology was *Tonvorstellung*.

Scholars have not agreed on the meaning of *Tonvorstellung*, and English speakers have not agreed on an appropriate translation. Tonal representation and tonal imagination are possibilities, but Riemann's meaning lies somewhere between the two: Representation is too abstract, and imagination too limiting. *Tonvorstellung* refers not only to one's imaginative capacity, but also to the results of using that capacity. Reading through a musical score is a case where musicians exert their tonal imaginations to produce tonal representations.

We can be sure of one thing that Riemann did not mean by *Tonvorstellung*. He specifically did not mean *Tonempfindung*, or the physiological sensation of tone that Helmholtz had examined a decade earlier. *Tonvorstellung* implied a higher level of mental activity than the level of mere sensation. By identifying music theory with this higher level, Riemann forced a distinction between psychoacoustics as practiced by Helmholtz and music psychology as proposed (one is to understand) in "Ueber das musikalische Hören."

The full weight of *Tonvorstellung* is not felt in Riemann's dissertation, but the importance of the concept is suggested several times. One detects a quasi-Kantian distinction between acoustical facts and their psychological representation. Overtones, beating, and so forth, belong to the physical world and are given to consciousness as *Vorstellungen*. Riemann did not articulate the correlation between physical and psychological reality, but exploited the difference to challenge empirical research that contradicted his beliefs about music. By recognizing distinct modes of mental and material existence, he was laying the groundwork for a psychological (as opposed to psychoacoustical) music theory. Chapter 1 of the dissertation presented the *Vorstellung* concept in its barest form: "The reaction of our mind (*Seele*) to a sensory impression is a *Vorstellung*: A light *Vorstellung* corresponds to a light stimulation, a heat *Vorstellung* to a heat stimulation, and a tone *Vorstellung* to a tone stimulation, etc."[48]

Riemann identified two classes of *Vorstellung*: simple and compound. He was concerned only with the latter and did not say what kind of stimulation would give rise to the former. Compound *Vorstellungen* could be similar or dissimilar. If they were similar, their similarity could be of two types: *Analogie* (analogy) or *partielle Gleichheit* (partial likeness). *Analogie* might involve the perception of timbre. For example, the bassoon notes E3–F3–F#3 all share a characteristic distribution of overtones even though the actual overtones for each pitch are different. Because the distribution is constant, the related *Vorstellungen* are analogous. *Partielle Gleichheit* on the other hand might involve the perception of two pitches an octave apart. Here the *Vorstellungen* would retain the psychological equivalent of common partials even if the pitches were played by different instruments. The "senario set" of C3 {C3, C4, G4, C5, E5, G5}, for example, intersects with that of C4 {C4, C5, G5, C6, E6, G6} no matter what instruments play C3 and C4. The intersection {C4, C5, G5} describes the partial likeness of the two *Vorstellungen*. Whereas timbre was constant and pitch varied in the bassoon example, here pitch is constant and timbre may vary. The constancy of pitch is not that strict—

48 Riemann, "Ueber das musikalische Hören," 7. [Die Reaction unserer Seele auf einen Sinneseindruck ist eine Vorstellung, und zwar entspricht einem Lichtreiz eine Lichtvorstellung, einem Wärmereiz eine Wärmevorstellung und einem Tonreiz eine Tonvorstellung, etc.]

C4 preserves just half the senario set of C3—but *partielle Gleichheit* requires that only some elements remain in common. Pitch does not vary enough in the C3–C4 example to call the resulting *Vorstellungen* dissimilar with respect to pitch. Pitch-dissimilar *Vorstellungen* would result from the bassoon example, since the senario sets there ({E3, E4, B4, E5, G#5, B5}, {F3, F4, C4, F5, A5, Ab5}, {F#3, F#4, C#4, F#5, A#5, C#5) intersect trivially as Ø.[49]

Unlike "Musikalische Logik," which includes a section on meter, "Ueber das musikalische Hören" deals only with harmony. *Tonvorstellung* is thus introduced and developed along harmonic lines, and remains peripheral in Riemann's later works on rhythm and meter (the term is ideally suited to both aspects of his work, since *Ton* means accent or stress as well as tone). Hauptmann is still Riemann's main influence, but his presence has diminished from what it was in "Musikalische Logik." Riemann now incorporates the ideas of Helmholtz and Oettingen, and introduces his own hypothesis of harmonic undertones.

The diversity of "Ueber das musikalische Hören" makes for good reading but is problematic from the standpoint of theory. Riemann insists on uniting the work of his precursors, but can find no combination of Hauptmann, Helmholtz, and Oettingen that lines up on the crucial issues

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[49] Where p and q are pitches and $S(p)$ and $S(q)$ are their senario sets, the senario set universe $P(p) = \{S(q) \mid S(q) \neq S(p) \text{ and } S(p) \cap S(q) \neq \varnothing\}$. $P(p)$ is the set of scenario sets that intersect ($\cap$) nontrivially with $S(p)$. For any given p, there are 22 q whose senario sets satisfy the definition of $P(p)$. $S(p)$ may intersect with itself (yielding identical *Vorstellungen* with respect to pitch) or disjunctly (yielding dissimilar *Vorstellungen* with respect to pitch), but total similarity and dissimilarity are trivial to Riemann's notion of *partielle Gleichheit*.

of dualism and harmonic function. Nor is there support among these theorists for the undertone series. As one would expect, the conflicts are most jarring in the case of minor harmony: Oettingen's logical dualism (see 2.4) explains the minor triad as a product of two overtone series, in effect, as a "back-formation" from the phonic overtone (G for harmonic series on C and E^b).[50] This is obviously different from Riemann's justification-by-undertones. It is also different from Hauptmann's quasi-dualism, which associates *having* with major and *being* with minor.[51] All three varieties of dualism—logical, quasi, and undertone—conflict with Helmholtz's explanation of minor harmony as a "getrübte Consonanz." These are not promising beginnings for a unified theory, and it is precisely the differences between Riemann and the theorists he wants most to absorb that cause such concepts as *Tonvorstellung* to splinter in different directions. Are *Tonvorstellungen* representations of acoustical, functional, or dualistic phenomena; or perhaps some combination of the three? One is not sure. Despite his best efforts at reconciliation, the intermingling of disparate sources in "Ueber das musikalische Logik" seeded tension in Riemann's work. Two sources of tension were the distinction between *Harmoniker* and *Kanoniker*, and the conflict between

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[50] In senario set terms $S(C2) \cap S(E^b2) = \{G4\}$.
[51] Hauptmann, 32–33. *Haben* is a positive or active state for Hauptmann, whereas *Sein* is a negative or passive state. When Hauptmann says that a pitch *is* a common overtone, he is saying something about the polarity of minor chords in general. Oettingen's misreading (p. 39) of Hauptmann undoes this web of associated meaning : "The tones of the major triad are common partials of a tonic fundamental; those of the minor triad have a common phonic overtone." [Die Töne des Durdreiklangs sind gemeinsame Bestandtheile eines tonischen Grundklanges, die des Molldreiklanges haben einen gemeinsamen phonischen Oberton.] Riemann's misreading of Oettingen undoes the *Haben-Sein* polarity altogether, since major and minor chords both *have* partial (under- or over-) tones.

function-as-category and harmonic dualism. We shall address these in turn.

3.10 Riemann and the *Kanoniker*

As we saw in Chapter 1, Riemann divided the history of music theory into two opposing strains, which he identified with Pythagorus on one hand and Aristoxenus on the other. The Pythagoreans had sought the essence of musical consonance in number, whereas the Aristoxeneans had denied an intrinsic relation between tone and number and elevated perceptual judgments instead. Riemann used the term *Kanoniker* in reference to the Pythagoreans, and *Harmoniker* in reference to the Aristoxenians.[52] He decried the tradition of *Kanoniker* from Pythagorus through Euler—"der letzte eigentliche Kanoniker"—and identified his own work with the tradition of *Harmoniker.*

In Riemann's view, four *Harmoniker* had done the most to move music theory beyond the "miraculous, secret counting" of *Kanoniker.* These were Hauptmann, Helmholtz, Oettingen, and Rameau. By any standards this was an odd quorum, and the tag *Harmoniker* did not entirely fit the facts. Number was of varying importance in the work of all these theorists. Indeed, Riemann's hard line of division conflicted with his own reliance on number throughout the dissertation.

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[52] Riemann was first to use *Harmoniker* in this sense, however *Kanoniker* was in use earlier; for example, in M. W. Drobisch, *Über musikalische Tonbestimmung und Temperatur* (Leipzig: Weidmann, 1852). The term *Kanoniker* refers to Euclid's "Division of the Canon" (κατατομη κανονος), and *Harmoniker* to Aristoxenus's "Harmonics" (ειςυγωγη αρμονικη).

Riemann began innocently enough by affixing numbers to the partial tones of the harmonic series on C2, to show their relative frequencies ("Grösse der Schwingungszahl"): C2 and 1, C3 and 2, G3 and 3... C5 and 8. He then assigned numbers to the partial tones of the undertone series, taking C6 as 1 and proceeding downward: C5 and $^1/_2$, F4 and $^1/_3$... C3 and $^1/_8$. The two series were symmetrically related and theoretically infinite, but Riemann stopped at 8 and $^1/_8$ because he believed that higher and lower partials were inaudible under normal conditions.

Riemann next tried to highlight the "chord-of-nature" by appealing more directly to number. The chord-of-nature had to be weeded out of the harmonic series because it was the basis of tonal harmony and Riemann's model of musical consonance. The weeding out process consisted of factoring the harmonic series and assigning greater importance to tones represented by prime numbers than to those represented by prime factors. Thus, 2 was more important than 4 (2^2) or 8 (2^3), 3 was more important than 6 ($2 \cdot 3$) or 9 (3^3), and so on. The result of Riemann's effort, through the first thirteen partials, is shown in Example 3-10 where "musical primes" are highlighted in bold type.

EXAMPLE 3-10: FACTORING THE HARMONIC SERIES

1, **2**, **3**, 2. 2, **5**, 2. 3, **7**, 2. 2. 2,
3. 3, 2. 5, **11**, 2. 2. 3, **13** etc.

Notice that Example 3-10 highlights the number 1, even though 1 does not belong to the set of prime numbers. Riemann may have considered 1 prime because it stood for the fundamental of the series. It would have made sense to consider this as the smallest musical prime, and to exclude 2 from the series since, harmonically speaking, 2 stands in the same relation to 1 as 4 does to 2. We might then speak of two sets of prime numbers—"numerical" (excluding 1) and "musical" (including 1, excluding 2). We could say, moreover, that the chord-of-nature has special prominence because of its correlation with the first three musical primes (1, 3, and 5). Or, keeping 1 for C2, we could say that the first three numerical primes give the chord-of-nature. Riemann can say neither since he starts from the fundamental and considers 1 and 2 both primes. It is enough for him that the chord-of-nature is established by the fifth partial, or fourth prime in Example 3-10.

Since factorization leaves behind all primes, and not just the ones corresponding to the chord-of-nature, number alone will not get Riemann what he wants. The weeding out establishes the chord-of-nature at the beginning of the prime series, but does nothing to rid Example 3-10 of the tones represented by 7, 11, and 13. In the cases of 11 and 13, Riemann simply asserted that the beating was too great for the intervals to be useful. He also rejected 7 but offered nothing in support of this except the failures of past attempts to treat the natural seventh as a consonance. The case of 7 deserved more care since by Riemann's time there was considerable disagreement over the status of the natural seventh. Riemann mentioned Kirnberger, Chladni, and Helmholtz as proponents of 7 (see 1.7) but did not elaborate their views.[53] Rameau, Hauptmann, and Oettingen on the other hand rejected 7, and believed as Riemann that the relations of music could be expressed by means of the smaller primes. Oddly enough, Riemann drew support for this notion from Euler, and thus blurred the distinction between *Kanoniker* and *Harmoniker*.[54]

Riemann's theory of consonance was induced directly by the numerical relations in Example 3-10. This theory was based on the notions of first-order and second-order overtones ("Obertöne erster und zweiter Ordnung"). First-order overtones stood in the relation of 2, 3, or 5 to a given fundamental. Such tones were not just closely related to the fundamental, they were directly related in the sense that no second-order partials stood between them and the generating tone. Higher primes possessed this attribute of direct relation too, but were ruled out for the reasons noted above. Riemann construed all remaining partials as either products or powers of first-degree overtones, thus the prime factors of Example 3-10. These so-called "Obertöne der Obertöne" made up the

[53] For a treatment of the number 7 in music theory see Martin Vogel, "Die Zahl Sieben in der spekulativen Musiktheorie," (Ph.D. diss., University of Bonn, 1954). Vogel is an advocate of the natural seventh 4 : 7, and has modified the Table of Relations to include a separate axis for seventh relations.

[54] Riemann, "Ueber das musikalische Hören," 16. On the importance of smaller primes Riemann writes: "It is known that all the acoustical relations of our music system can be expressed by means of products and powers of the numbers 1, 2, 3, and 5; as Euler once said, following Leibniz's dictum: in music...it is not customary to count beyond five." [Es ist bekannt, dass sich alle akustischen Verhältnisse unseres Musiksystems durch Producte und Potenzen der Zahlen 1, 2, 3 und 5 ausdrücken lassen; wie Euler dem Leibniz die Worte in den Mund legt: in musica... ultra quinarium numerari non soleri.] The Euler reference is from *Tentamen*, chap. 10, §19, and recalls Leibniz's famous definition of music as an unconscious exercise by a mind that doesn't know it is counting ("exercitium arithmeticae occultum nescientis se numerare animi"). Riemann was unaware that Euler reconsidered and decided in *Du véritable caractère de la musique moderne* (1764) that musicians had in effect learned to count to seven.

larger field of second-order overtones. Consonant relations were determined by first-order overtones 2, 3, and 5, but strictly speaking not by these alone. Because Riemann viewed 1 as a kind of musical prime, 2, 3, and 5 together with 1 formed his archetype for consonance, which he called the *Primklang*—primary in the general sense but also in the specific sense of being derived from *Primzählen*. Tones that belonged to one and the same *Primklang* were consonant, and tones that belonged to different *Primklänge* were dissonant. We have thus far ignored questions of octave equivalence and minor consonance in connection with the *Primklang*. At first Riemann seemed to take "gehören" literally to mean that membership included only 1, 2, 3, and 5, and excluded octave multiples such as $2 \cdot 2$, $2 \cdot 3$, and $2 \cdot 5$. His statement that only those tones "are consonant with the fundamental that cannot be related more closely to another tone of the series" affirmed the situation of Example 3-10.[55] Under this interpretation, however, it is impossible to produce a consonant major chord in close position (or in different inversions for that matter). Partials 4, 5, and 6 give such a chord, but as Example 3-10 shows involve the compounds 4 and 6 which by definition are second order. "Belonging to a *Primklang*" was thus modified to mean 1, 3, 5 and their octave multiples. Riemann gave the following formula to clarifiy this more inclusive notion of consonance: $1 \cdot 2^m : 3 \cdot 2^n : 5 \cdot 2^t$ (where m, n, and t are integers). His definition of dissonance was correspondingly narrowed to "the simultaneous sounding of a secondary overtone (which is not the octave of a primary overtone) with the main tone."[56]

55 Riemann, "Ueber das musikalisches Hören," 17. [die nicht näher auf einen anderen Ton der Reihe als den Grundton bezogen werden können, sind mit ihm consonant.]
56 Riemann, "Ueber das musikalische Hören," 22. [das gleichzeitige Angeben eines secundären Obertönes (der nicht Octav eines primären ist)... mit dem Haupttone.]

Riemann's concession to octave equivalence is inconsistent with the irreducibility-of-primes idea behind his theory of consonance. If powers of 2 are equivalent, why not powers or 3 or 5? Musically the answer may seem obvious, but numerically it does not.

Minor consonance was developed analogously to major consonance: Riemann factored the undertone series and presented an inverted version (*Gegenbild*) of Example 3-10. Here the *Unterprimklang* was identified with first-order undertones 1, $^1/_3$, and $^1/_5$, and minor consonance was given by the formula: $1 \cdot 2^m : {}^1/_3 \cdot 2^n : {}^1/_5 \cdot 2^t$ (where m, n, and t are integers). Riemann argued that the minor chord was equal in consonance to the major chord because in point of derivation it was simply an inversion of that chord. This was not the old dualism of Oettingen. Riemann's dualism-by-undertones entailed a new view of the *Klang*, one in which major and minor chords were the simultaneous products of a common fundamental (see n. 51). There was a sense in which the resulting chords (e.g. F–A♭–C, C–E–G) were components of a single chord, a fact that helps explain the strength Riemann later assigned T–°S progressions (the *Seittonwechsel*) in his function theory. We should note, however, that Riemann derived minor harmony after major harmony: The *Unterprimklang* emerged in the shadow of the *Primklang* and, generally, the minor mode was accorded little more attention in "Ueber das musikalische Hören" than in "Musikalische Logik."

Riemann's factoring method was unsatisfactory because it dispensed with higher primes arbitrarily, and then conceded that octave multiples of the lower primes represented intervallic consonances after all. A better method would have started with 2, 3, and 5, and given some rationale for preferring their compounds. A framework in which the compounds were shown as salient, in other words, would have given cause to believe in the salience of the primes. We consider one possible framework in Example 3-11.

The model we shall use is called a sieve and comes from a branch of number theory that is concerned with sifting prime numbers from composites. Many sieves or sieve-like procedures have been developed to catch primes, but none so far has been able to catch all primes. The sieve we shall use—one of the oldest and best known—is the sieve of Eratosthenes (ca. 276–ca. 194 B.C.), named for the Greek astronomer who discovered it. Example 3-11a presents the sieve of Eratosthenes in a tabular format.[57]

[57] See Sir Thomas L. Heath, *A Manual of Greek Mathematics* (New York: Dover, 1963), 63; and Ivars Peterson, *The Mathematical Tourist: Snapshots of Modern Mathematics* (New York: W. H. Freeman, 1988), 21–23.

EXAMPLE 3-11: SIEVE OF ERATOSTHENES AND MUSICAL PRIMES

ERATOSTHENES SIEVE AND MUSICAL PRIMES

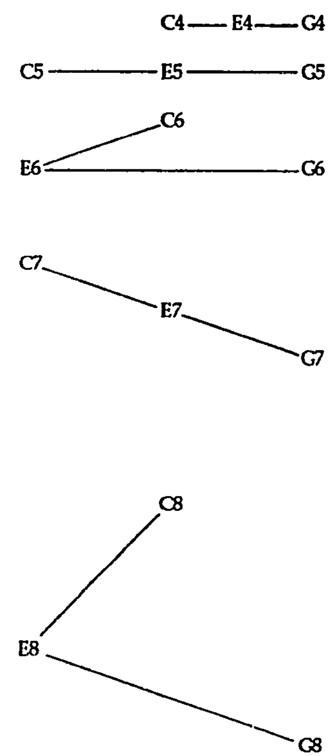
A: The Sieve

1	[2]	[3]	4	[5]	6
[7]	8	9	10	[11]	12
[13]	14	15	16	[17]	18
[19]	20	21	22	[23]	24
25	26	27	28	[29]	30
[31]	32	33	34	35	36
[37]	38	39	40	[41]	42
[43]	44	45	46	[47]	48
49	50	51	52	[53]	54
55	56	57	58	[59]	60
[61]	62	63	64	65	66
[67]	68	69	70	[71]	72
[73]	74	75	76	77	78
[79]	80	81	82	[83]	84
85	86	87	88	[89]	90
91	92	93	94	95	96
[97]	98	99	100		

B: Musical Primes

C2	[C3]	[G3]	C4	[E4]	G4
[Bb4]	C5	D5	E5	[F5]	G5
[A5]	Bb5	B6	C6	[C#6]	D6
[Eb6]	E6	F6	F#6	[•]	G6
G#6				[•]	
[•]	C7				
[•]			E7	[•]	
[•]				[•]	G7
				[•]	
				[•]	
[•]			C8		
[•]				[•]	
[•]					
[•]	E8			[•]	
				[•]	
					G8
[•]					

C: Plotting the 'Chord of Nature'



We have limited the sieve's range to numbers 1 through 100, and ordered these consecutively over six columns. This six-to-a-row ordering allows for an easy determination of primes by using the following method: Beneath the first prime "2" a vertical line strikes out all multiples of 2. Beneath the second prime "3" a vertical line strikes out all multiples of 3. Column four is struck entirely because its numbers are all multiples of 2. The same occurs in column six. From "5" a series of diagonals strikes out multiples of 5, and from "7" a series of diagonals strikes out multiples of 7. At this point our work is done and the twenty-five remaining numbers [in square brackets] are the set of primes between 1 and 100.

Example 3-11b interprets the numbers of 3-11a as partial tones, beginning with C2 = 1. Thus, the first row of 3-11a becomes the senario set of C2. The translation from number to pitch nomenclature is rough and incomplete. By the seventh partial [B^b4] the correspondence between letter name and pitch is shaky;[58] by the twenty-fifth [$G^{\#}6$] it is difficult to go on without elaborate notations for pitches other than C, E, and G. From the twenty-sixth to one-hundredth partial, therefore, we list only octave multiples of 2, 3, and 5. Asterisks in square brackets serve as place holders

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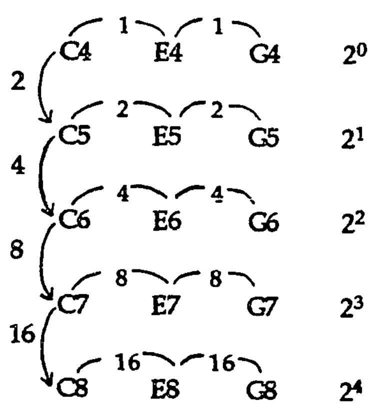
58 We take the equal-tempered scale to be normative: The third and fifth partials differ by only 2 cents and 14 cents, respectively, from their equal-tempered counterparts; the seventh partial differs by 31 cents (well over a comma), and the eleventh by 49 cents (almost a quartertone). Ellis (p. 456) calls the natural seventh "subminor." The ratio of fundamental to subminor seventh is 4 : 7, and the pitch is 969 cents from the fundamental. This is 27 cents lower than the B^b (996 cents) 2Q west of C, and 49 cents lower than the B^b (1018 cents) -T + 2Q of C. This latter "Knight's move" B^b is itself a syntonic comma (21.5 cents) higher than the western B^b. The term "Knight's move" is borrowed from Lewin (1987), to refer to compound moves consisting of one major third (T) and two perfect fifths (2Q). Such moves, considered in relation to the Table, are geometrically equivalent to a Knight's move on a chessboard. There are four Knight's moves altogether: T + 2Q, -T + 2Q, T + -2Q, -T + -2Q. Notice that these moves are commutative; the order in which T/-T and 2Q/-2Q are combined has no effect on the outcome: T + 2Q = 2Q + T.

for the larger primes of Example 3-11a. Example 3-11c reduces things further by plotting only the chord-of-nature in four octaves. Here we omit the entire first column of Examples 3-11a and 3-11b and skip primes 2 and 3, so that we begin with the close-position compound C4–E4–G4. By giving the octave multiples of this compound, we begin to see the pattern formed by multiples of 2, 3, and 5. Example 3-12 makes this pattern explicit. The chord-of-nature is distributed exponentially (on base 2) throughout the sieve: C4–E4/E4–G4 are adjacent, or one place (2^0) apart in the sieve; C5–E5/E5–G5 are two places (2^1) apart; C6–E6/E6–G6 are four places (2^2) apart; C7–E7/E7–G7 are eight places (2^3) apart; C8–E8/E8–G8 are sixteen places (2^4) apart. The chords themselves are distributed similarly: C5–E5–G5 appears two places (2^1) after G4; C6–E6–G6 appears four places (2^2) after G5; C7–E7–G7 appears eight places (2^3) after G6; C8–E8–G8 appears sixteen places (2^4) after G7. By this means we ascribe significance to octave multiples of 2, 3, and 5, and suggest a numerical rationale for the major chord represented by the first three primes.

EXAMPLE 3-12: EXPONENTIAL DISTRIBUTION OF THE CHORD-OF-NATURE



3.11 Paradigms in Tension

Earlier we spoke of the *große Cadenz* as paradigm in "Musikalische Logik." We explained that the categorical approach to function developed there was incompatible with Oettingen's dualistic theory of chord connection. The broader point was that Hauptmann's abstract conception, which informed Riemann's initial formulations, was incompatible with Oettingen's concrete conception. We noted further that the paradigm for this latter conception was the Table of Relations. The tension between abstract and concrete was not critical in "Musikalische Logik." It became critical in "Ueber das musikalische Hören," where Riemann entertained both paradigms, backing away from the old (but not letting go) and approaching the new (but not embracing it).

In preparing his dissertation, Riemann combed "Musikalische Logik" for occurrences of Hauptmann's *Einheitsbegriff*, *Quintbegriff*, and *Terzeinigung*, and substituted his own *These*, *Antithese*, and *Synthese*.[59] A related change was the absence of bracket notation, which Riemann had used before to present chords as abstract *Begriffe*. The introduction of the Table was thus one of several small novelties which, taken together, suggested a reassessment of earlier ideas. This reassessment was perhaps out of concern for the minor mode, which had no logic in the old paradigm, and almost certainly spurred by Riemann's discovery of Oettingen's *Harmoniesystem*. Still, the *große Cadenz* persisted and minor-mode progressions (as well as the Table itself) received marginal

[59] Seidel (pp. 48–49) shows parallel passages from the two works in which these substitutions occur.

attention in the new work. One might say that Riemann was in a "pre-paradigm period" with respect to the Table.[60] He was interested in it, but felt no consequent pressure to give up the *große Cadenz*. This pull between Hauptmann and Oettingen was long-standing, and one could argue that Riemann never completely exorcised Hauptmann from his work. Our view is that the Table was the mise-en-scène for Riemann's struggle with his precursors, that it eventually displaced the *große Cadenz*, and that this displacement of paradigms profoundly affected the development of the concept of *Tonvorstellung*.

3.12 The Table of Relations: A Footnote

The Table of Relations appeared in chapter 3 ("Tonalität in Dur und Moll") of the dissertation, where Riemann elaborated his conception of tonality. He saw fit to divide his treatment of this topic into sections on memory, tonal connection, intervallic inversion, musical logic, and melody in both major (*Durgeschlecht*) and minor modes (*Untertongeschlecht*). In the section on chordal connection (*Klangverknüpfung*), Riemann broke off and introduced the Table in an eleven-page footnote treating pitch and chord intonation. The length of the footnote—roughly one-sixth that of the dissertation—showed Riemann's preoccupation with tuning. More than half of it consisted of lists of available tunings for each of the twelve letter names of the chromatic scale. His interest was not restricted to tuning, however, for

[60] The term comes from Thomas Kuhn, *The Structure of Scientific Revolutions* 2d ed. (Chicago: Univ. of Chicago Press, 1970), 96.

tuning differences were concomitant with differences of function. Tuning demarcated the Table, in other words, so that functional differences—the difference between the "western B^{b}" and the "Knight's move B^{b}"—were made explicit. This explicitness was lost in Riemann's subsequent espousal of equal temperament. Once acoustical differences were wiped away from the Table, function became a straightforward matter of geometry. In Example 3-13, however, we present the Table as it appeared in "Ueber das musikalische Hören."

EXAMPLE 3-13: TABLE OF RELATIONS FROM "UEBER DAS MUSIKALISCHE HÖREN"

5	4	3	2	1	0	1	2	3	4	
Cis	Gis	dis	ais	eis'	his'	fisis"	cisis'"	gisis'"	disis'"	3
,A	E	H	fis	cis'	gis'	dis"	ais"	eis'"	his'"	2
,F	C	G	d	a	e'	h'	fis"	cis'"	gis'"	1
,Des	,As	Es	B	f	c'	g'	d"	a"	e'"	0
„Bb	,Fes	Ces	Ges	des	as	es'	b'	f'	c'"	1
„G♭♭	,D♭♭	,A♭♭	E♭♭	Bb	fes	ces'	ges'	des"	as"	2
„E♭♭♭	„B♭♭	,Feses	Ceses	Geses	deses	asas	eses'	heses'	fes"	3
5	4	3	2	1	0	1	2	3	4	

Notation is the first order of business. Riemann gave the impression that he was quoting "einer von A. v. Oettingen entworfenen Verwandschaftstabelle," but his Table differed considerably from Oettingen's with respect to notation.[61] Upper *Striche* accompanied lower-

[61] See 2.6 of the present study.

third relations in Riemann's Table, and lower *Striche* accompanied upper-third relations. The reverse was true of Oettingen's Table. Riemann also used uppercase and lowercase *Buchstaben*—along with superscripts and subscripts—to keep track of octave displacements where they occurred. Oettingen used unscripted *Buchstaben*, all in lowercase, and did not track octave displacements. These differences seem small but were not trivial. Oettingen emphasized the modularity of the Table by using generic letter names: The central 'c' stood equally for all Cs. Oettingen's *Striche* emphasized the form of the Table: Apart from their intonational significance, they served as markers showing distance (in pure thirds) and direction ("north" or "south") from the central series. Riemann on the other hand used pitch-specific letter names (c' = middle C = C4), and emphasized the intonational significance of *Striche*.[62] A lower

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[62] Riemann claimed inaccurately that the *Buchstabentonschrift* of his Table "ist die von A. v. Oettingen erfundene mit den von Helmholtz in der 3. Aufl. der L. v. T. [1870] gemachten Veränderungen" (p. 29). We should take a moment to outline the somewhat confusing development of this notation, since it became standard in nineteenth-century treatments of tuning and intonation. Hauptmann (1853) used uppercase and lowercase letters to distinguish fifth and third relations: 'e' (third of 'C') represented a pitch one syntonic comma lower than 'E' (fourth fifth of 'C'), C–e–G represented the C-major *Hauptklang*, and F–a–C–e–G–b–D the C-major tonality. This notation was inadequate for representing deviations by two or more commas, and was modified by Helmholtz in the first edition (1863) of *Tonempfindungen*. Helmholtz introduced *Striche* for deviations involving two commas, so that '$\underline{G}^{\#}$' (third of 'e') was one comma less than '$g^{\#}$' (third of 'E') and two commas less than '$G^{\#}$' (eighth fifth of 'C'); '$\overline{F}^{bb}$' (under-third of 'a^{b}') was thus one comma greater than 'f^{b}' (under-third of 'A^{b}') and two commas greater than 'F^{b}' (eighth under-fifth of 'C'). Oettingen (1866) omitted uppercase letters altogether, so that number–of–*Striche* corresponded to number–of–commas, and used upper *Striche* for upper thirds and lower *Striche* for lower thirds. Helmholtz abandoned uppercase letters in the third edition (1870) of *Tonempfindungen*, but retained upper *Striche* for lower thirds and lower *Striche* for upper thirds. Riemann followed Helmholtz in this respect, but otherwise followed the notational conventions for the "small" and "great" octaves: c – b = small octave (C3–B3); C – B = great octave (C2–B2); c'–b' = C4–B4; ,C–,B = C1–B1; c''–b'' = C5–B5, and so on. The use of *Striche* is confused by two further factors. As early as Euler they were used to show octave displacements instead of tuning distinctions, and thus replaced accents in notations such as c'–b' or ,C–,B. For such usage, see Drobisch, 90–91. Finally, Helmholtz's *Striche* were omitted in Ellis's widely consulted translation (1885) of the fourth edition (1877) of *Tonempfindungen*. Ellis introduced his own notation, which readers should compare with the *Striche* in the third and fourth edition of *Tonempfindung*.

Strich in Riemann denotes a lowering of pitch, and is not suggestive of direction. A lower *Strich* in Oettingen points south, and is not suggestive of pitch. We may summarize by saying that Oettingen favored a general notation oriented toward form (of the Table), whereas Riemann favored a specific notation oriented toward pitch.[63] Oettingen's notation was appropriate to his logical conception of dualism, whereas Riemann's was appropriate to his acoustical conception.

The presence of the Table in a section on *Klangverknüpfung* is of some importance. *Verknüpfung* is one of several German words for "connection." It indicates a particularly close connection, much closer than the commonly used *Verbindung* would describe. When applied to thought, *Verknüpfung* indicates a logical connection (*logisch verknüpfte Gedanken*), whereas *Verbindung* would suggest a casual association.[64] *Klangverknüpfung* in turn indicates a systemic connection between tones or chords. The term would have suited Oettingen's system of chord connection perfectly (Oettingen used the less suggestive *Klangfolge*). Riemann understood *Klangverknüpfung* not in Oettingen's chord-transformational sense—at least not yet—but in a psychological sense. *Klangverknüpfungen* were the mental connections that forged partial or

[63] This difference persisted in Riemann's appropriation of chord symbols and terminology from Oettingen. Riemann invariably complicated Oettingen's ideas by adding to them; sometimes the extra precision paid off, but often it did not (see Chap. 2 n. 17, concerning Riemann's homologic-antilogic terminology).
[64] See R. B. Farrell, *A Dictionary of German Synonyms* (London: Cambridge Univ. Press, 1968), 173.

analogous relationships between *Tonvorstellungen*. Such connections were logical in the same way as the *große Cadenz*, namely, as manifestations of human hearing (which Riemann later described as a "a highly developed operation of logical functions of the human mind").[65] Riemann used the term *Klangverknüpfung* exclusively to underscore this psycho-logical view of tonal relation. When he spoke generically of chord relations, he used the more conventional *Klangfolge* or *Fortschreitung*. Because the Table was presented in the context of a discussion of *Klangverknüpfung*, and not *Klangfolge* or *Fortschreitung*, one suspects that it already had some psychological significance for Riemann.

Verknüpfungen are most easily made between pitches an octave apart, since their respective *Vorstellungen* will have several partials in common. Riemann invoked an acoustical rationale based on the loudness or *Intensität* of partial tones to explain the ease of such connections. This rationale was problematic because it assumed that the amplitude of partial tones was independent of pitch and timbre.[66] Riemann's method for determining the *Verknüpfung* potential of two pitches was roughly as follows: First, he considered the pitches not as independent *Klänge* (complex tones), but as harmonically related partial tones. Next, he calculated the *Intensität* of the higher partial by squaring the inverse of its frequency relative to the lower partial. Finally, he observed the difference between this *Intensität*—which was some fraction

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[65] Riemann, "Ideen," 1. [hochgradig Betätigung von logischen Funktionen des menschlichen Geistes.]

[66] Riemann may have misread Helmholtz, who gave conditions under which the amplitude of some partial number n decreased constantly as $1/n$. Under different conditions, however, this amplitude could decrease as $1/n^2$ or even as $1/n^3$. See Helmholtz, Ellis trans., 35n.

of 1—and 1 itself, which would be the *Intensität* of the upper partial once it was reinstated as fundamental of its own chord. The smaller the difference, or greater the *Intensität* of the higher partial, the easier the connection and more closely related the tones. To observe this theory in practice, we consider connections between C2 and each of the pitches belonging to its *Primklang*, namely C3, G3, and E4. We begin with the understanding that C2 and C3 are not Klänge, but partial tones; specifically, that C2 is the fundamental and C3 its first upper partial. The frequency of C3 is twice ($^2/_1$) that of C2. We therefore square the inverse ($^1/_2$) of this frequency to determine that C3 is $^1/_4$ as loud as C2. Finally, we observe that in the progression C2–C3 the *Intensität* of C3 rises from $^1/_4$ to 1, which reflects the fact that C3 occurred first as an upper partial where it was $^1/_4$ as loud as C2, and then as a fundamental in its own right. We determine similarly that G3 is $^1/_9$ as loud as C2, and E4 $^1/_{25}$ as loud. In the three progressions C2–C3, C2–G3, and C2–E4 the increase of *Intensität* is respectively, $^1/_4$ to 1, $^1/_9$ to 1, and $^1/_{25}$ to 1. Because the difference between these numbers gets larger, we conclude that C3, G3, and E4 are progressively more difficult to hear in connection with C2.

The above calculations agree generally if not exactly with the facts. Since lower partials are louder than higher ones, Riemann reasoned that they persisted longer in memory and thus facilitated *Klangverknüpfung*. Distantly related *Klänge* were distant because their common partials were imperceptible, or (what amounted to the same thing) too transient to serve as a basis for comparison. Riemann's assumption was that *Tonvorstellungen* lingered in memory, but were highly volatile, and that the fading away (*Abklingen*) of these structures corresponded to the

Intensität of their various partials.[67] Some connections were obviously more feasible than others. In cases involving second-order overtones, for instance, the connections became quite tenuous. The progression C2–D5 (the ninth partial of C2, or third partial of G3) was for all purposes dissonant, since D5—with just $^1/_{81}$ the *Intensität* of C2—would have faded from the C2-*Vorstellung* long before recurring as fundamental of the D5-*Vorstellung*. The situation worsened as octave compounds were brought into close position. Since C5 was $^1/_{64}$ the *Intensität* of C2, D5 was just $^1/_{64} \cdot {}^1/_{81}$ that of C5. Close position had a dramatic effect on *Verknüpfung* involving musical primes as well: E4 had just $^1/_{16} \cdot {}^1/_{25}$ the *Intensität* of C4 (relative to C2), and the progression C4–E4 entailed a change of *Intensität*—$^1/_{400}$ to 1—that seemed to exceed any realistic limits of comprehension. Since comprehensible relations held only among components of the *Primklang*, C4–E4 was at best a borderline consonance (because C4 is represented by a composite number). The example is illustrative of a larger issue concerning major third relations. Riemann recognized that even the fifth partial—at just $^1/_{25}$ the *Intensität* of the fundamental—was quite weak. This weakness led him to pronounce, in general, that third relations were weaker than fifth relations. This was not a position Riemann would maintain; for the time being however, he claimed that 5 was rarely used as a connective between harmonies (see the discussion of Example 3-17 below), and that unmediated progressions of a major third sounded somewhat surprising. We may assume from this that Riemann construed the Table horizontally, in the manner of

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[67] Riemann, "Ueber das musikalische Hören," 27. Riemann credits the idea of memory as *Abklingen* to Eduard von Hartmann, whom he calls "Philosoph des Unbewussten."

Oettingen, as consisting of third-related series of fifths rather than fifth-related series of thirds. (Riemann's Table does include more remote relations above and below the central series, though.)

Riemann gave no indication of knowing Euler's mirror, or indeed any of that mathematician's work outside of *Tentamen.* It was a reference to Euler, nonetheless, that directly preceded Riemann's discussion of the Table. Riemann remarked on the fact, long since declared by Euler, that the further along an interval was in the harmonic series the more it deviated from acoustical purity. This fact was apparently the *causa causans* of a musical practice in which performers, intent on neutralising the effects of such deviation, raised the leading tone so that it formed an interval closer to a Pythagorean third with the dominant (64 : 81 or 408 cents), than a just third (4 : 5 or 386 cents). The same practice was behind the insistent interpretation of C♯s higher than D♭s, F♯s higher than G♭s, and so on. Riemann hoped to illustrate by means of the Table that one and "derselbe Ton" (he surely meant "letter name") could range in meaning, and that each meaning would entail a unique intonation. To this end, he presented a "Verzeichniss aller innerhalb eines Tonstückes möglichen Werthe,"[68] consisting of 133 possible values spread over the letter names of the chromatic scale—almost twice as many as those represented on the Table in Example 3-13. He gave these values in a series of twelve lists, each corresponding to one of the keys (*Taste*) of the equal-tempered keyboard. Example 3-14 reproduces the first of these lists, the set of values for C.

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68 Riemann, "Ueber das musikalische Hören," 30.

EXAMPLE 3-14: INTONATION VALUES FOR *TASTE* C

Taste c gleichschw. Temperatur. = 0,000000:

$\underline{\underline{\text{c}}}$	= 0,964160 — 1
$\underline{\underline{\underline{\text{his}}}}$	= 0,965784 — 1
deses	= 0,980456 — 1
$\underline{\text{c}}$	= 0,982080 — 1
$\underline{\underline{\text{his}}}$	= 0,983704 — 1
$\overline{\text{deses}}$	= 0,998376 — 1
c	= 0,000000
$\underline{\text{his}}$	= 0,001624 Schisma = 32805 : 32768.
$\overline{\overline{\text{deses}}}$	= 0,016296 Diaschisma = 2048 : 2025.
$\overline{\text{c}}$	= 0,017920 synt. Komma = 81 : 80.
his	= 0,019544 pyth. Komma = 531441 : 524288.
$\overline{\overline{\overline{\text{deses}}}}$	= 0,034216 kl. Diesis = 128 : 125.
$\overline{\overline{\text{c}}}$	= 0,035840 = $(81:80)^2$ = 6561 : 6400.
$\overline{\overline{\overline{\overline{\text{deses}}}}}$	= 0,052136 gr. Diesis = 648 : 625.

Riemann used logarithms on base 2 to show differences of pitch in Example 3-14. A logarithm is an exponent that indicates the power to which a number (called the base) must be raised to yield some other number. Any number may serve as base but 10 is particularly common: the logarithm of 100 on base 10 is 2 ($\log_{10} 100 = 2$), since $10^2 = 100$. By comparison, the logarithm of 100 on base 2 is 6.643856 ($\log_2 100 = 6.643856$), since $2^{6.643856} = 100$. Logarithms are useful for describing musical intervals because they can be added or subtracted to give the logarithmic values of other intervals. This is an advantage when one is working with complex intervals, since it is much easier to add or subtract integers than it is to multiply or divide ratios. Ratios are also harder to grasp intuitively; it is not obvious which of 32768 : 32805 and 531441 : 524288 is the larger interval until one determines the base 2 logarithm for each and sees that the first (.001628) is in fact smaller than the second (.019550) by a syntonic comma.[69] Base 2 logarithms are the preferred "musical logarithms" because $\log_2 2$ (the octave) = 1, which means that intervals within the octave are assigned a decimal place between 0 and 1, and octave multiples are represented by whole numbers: C4 = 0, C5 = 1, C6 = 2, and so on. By comparison, $\log_{10} 2 = .30103$ gives the less intuitive values of C4 = 0, C5 = .30103, C6 = .60206, and so on.[70] The logarithms in Example 3-14 are accurate to six decimal places. Notice that Riemann also included the ratios and traditional names for intervals of historical interest (schisma, diaschisma, comma).

Example 3-14 is further evidence of the *Kanoniker* influence in Riemann's early work. Riemann even acknowledged a debt to Drobisch, who used musical logarithms in his *Über musikalische Tonbestimmung und Temperatur* (1852), and to Euler, who was believed to have originated the practice.[71] While Riemann claimed he was opposed to the assumptions of both *Kanoniker* (and criticized Drobisch's analogies between tone and color, as Helmholtz had done earlier in his *Handbuch der physiologischen Optik*),[72] the seven pages of logarithmic values could

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69 32768 : 32805 is the so-called schisma (about 2 cents) between central c and b$^{\#}$ (8Q + T from c); 531441 : 524288 is the Pythagorean comma (about 23.5 cents) between central c and b$^{\#}$ (12Q east of c). The difference between the Pythagorean comma and the schisma is the syntonic comma.

70 Ellis's "Table of Intervals" uses base 10 logarithms, along with musical cents and traditional ratios. See Helmholtz, Ellis trans., 453–56.

71 This belief has persisted into the twentieth century. See Guy Warrack, "Music and Mathematics," *Music and Letters* 24 (1945): 22. Warrack credits Euler with the first use of musical logarithms. According to Barbour, however, musical logarithms were first used by Bishop Juan Caramuel de Lobkowitz (1606–82) in his *Mathesis nova* (1670), and again by Christian Huygens (1629–95) in his "Novus cyclus harmonicus" (1724). See J. M. Barbour, "Musical Logarithms," *Scripta Mathematica* 7 (1940): 21–31.

72 Drobisch draws an analogy between tone and color in appendix 2 of *Über musikalische Tonbestimmung und Temperatur*. Intonation models resembling the Table appear in appendix 1, and *Striche* are used on pp. 89–91 to denote octave relations. Riemann's attribution of this work to "F. W." rather than "M. W." Drobisch is curious, given that

have come directly from a work of Drobisch or Euler. Indeed, chapter 10 of *Tentamen* contained similar listings of musical logarithms.

Riemann treated chord relations only once in reference to the Table in "Ueber das musikalische Hören." To determine whether C# should be taken higher than D♭, he observed that 'c#' 7Q of center (2048 : 2187; 114 cents) was higher than 'd♭' -5Q of center (243 : 256; 90 cents) by about an eighth of a tone (the difference of 24 cents divides into a whole tone of 204 cents exactly 8.5 times), but that all remaining values of D♭—on the Table and in the lists—were higher than C#. He then cited a chord progression that contained both C# and D♭, and argued that context rather than an arbitrary rule should determine the function and relative intonation of the pitches. We reproduce this progression below in Example 3-15.

EXAMPLE 3-15: CHORD RELATIONS AND THE TABLE

Riemann interpreted the first two chords of Example 3-15 as belonging to F minor (b♭–d̄♭–f–ā♭–c–e̲–g) or, alternatively, to Hauptmann's F minor-

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Drobisch was one of the committee members who had rejected his inaugural dissertation in Leipzig. If "F. W." was a typographical error, it was perpetuated three times in the introduction to "Ueber das musicalische Hören," and left unchanged in the editions by F. Andrä (1874) and C. F. Kahnt (1874). Helmholtz's view of the tone-color analogy appears in *Handbuch der physiologischen Optik*, vol. 2 (Hamburg und Leipzig: L. Voss, 1867), 237.

major (b^{b}–$\bar{d}^{b}$–f–$\underline{a}$–c–$\underline{e}$–g).[73] Between chords 4 and 5, however, a modulation to G minor (or G minor-major) occurred, and the intonation of the C# would determine whether this modulation was to $\underline{g}$–b^{b}–$\underline{d}$ or to g–$\bar{b}^{b}$–d. Riemann offered two readings of the C# in chord 4.[74]

One first had to decide whether the modulation at chord 4 was chromatic or diatonic. Chromatic modulations involved *Leittonschritte,* whereas diatonic modulations involved *Wechsel.* Riemann was not clear about modulation by *Leittonschritt,* but seemed to mean pivot-tone modulations where a leading tone served as pivot and the remaining voices moved chromatically. Modulation by *Wechsel* involved a simpler "change" in the meaning of some diatonic pitch, typically an element of the *Primklang.* The modulation in Example 3-15 was diatonic because of the common (*verbindende*) tones E–G in chords 3 and 4, and because the melodic *Leittonschritt* D^{b}–C was followed in chord 4 by C# instead of the *Leitton* D^{b}. The intonation of D^{b} in chord 2, '$\bar{d}^{b}$' (112 cents), was given by its function as leading tone in F minor. The intonation of C# in chord 4 could not be the same as D^{b} because its function was not the same. The determination of its exact function depended on which *verbindende* tone was held in common between chords 3 and 4 (the Table's geometry does not allow '$\underline{e}$' and 'g' to function simultaneously as common tones). Riemann first speculated that '$\underline{e}$' (T of 'c') was retained in chord 4, to become Q of '$\underline{a}$', with the result that 'g' became '$\underline{g}$' and C# became T of '$\underline{a}$', or '$\underline{\underline{c}}^{\#}$' (70 cents). Under this interpretation, shown in Example 3-16, the C# of

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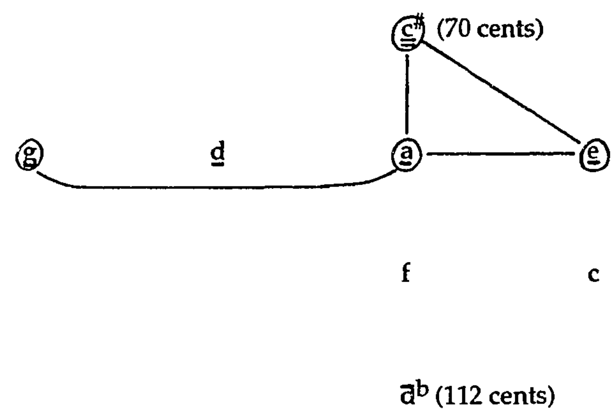
[73] See Hauptmann, 39–40.

[74] The modulatory pattern is one in which tonic harmony is circumscribed (here by two dominants) but never stated. Modulation by circumscription (*Umschriebung*) crops up in later discussions of the Table. See the discussion of "system steps" later in this chapter.

chord 4 was 3T north of the original D^b (112 cents), and about a fifth of a tone lower (42 cents). The four pitches of chord 4 are circled below.

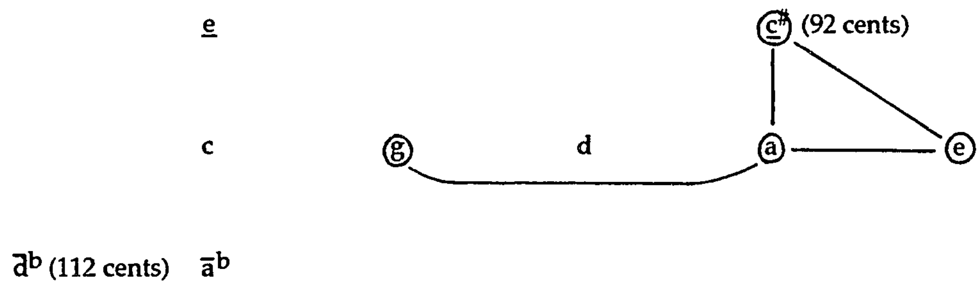
EXAMPLE 3-16: CHORD 4 WITH "E" AS COMMON TONE



If one continued the modulatory sequence according to the above interpretation, one would trace a northwest path into increasingly remote regions of the Table. Such a path would be an example of what Riemann called "vague" modulation, because it involved a departure from the central series.

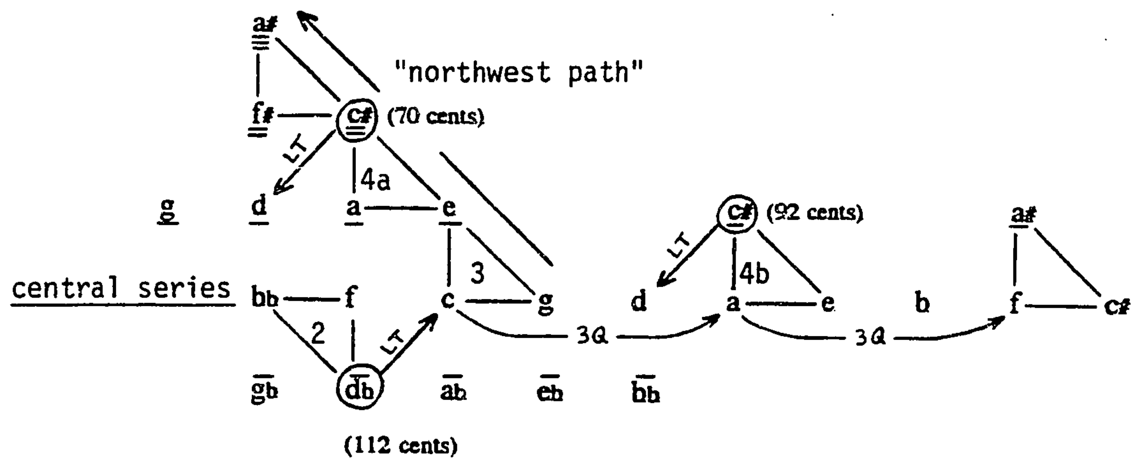
A simpler analysis of Example 3-15 was to retain 'g' as the common tone, in which case chord 4 was exactly 3Q east of chord 3 with 'g' as its seventh. Under this interpretation, shown in Example 3-17, $C^{\#}$ became T of 'a', or '$\underline{c}^{\#}$' (92 cents), and was just 20 cents lower than the original D^b. The modulation was no longer vague because there were no vertical departures from the central series. If one continued the sequence, one would move further east—3Q at a time—but never jump to a new series. The four pitches of chord 4 are again circled.

EXAMPLE 3-17: CHORD 4 WITH "G" AS COMMON TONE



Example 3-18 consolidates the analyses of Examples 3-16 and 3-17. The "vague" modulation of Example 3-16 is shown on the left side, with an arrow indicating the northwest path, and modulation by 3Q is shown on the right, with arrows beneath the central series. Chord 4 is labeled twice: 4a for the northwest reading, and 4b for the 3Q reading. Chords 2 and 3 of Example 3-15 are labelled 2 and 3. Finally, we show that the two C#s (70 cents; 92 cents) as well as the original Db (112 cents), function as leading tones, but to different pitches. These leading-tone functions are labeled LT, and the leading tones themselves are circled.

EXAMPLE 3-18: "VAGUE" MODULATION VERSUS MODULATION BY 3Q



Riemann managed in the course of his analysis to give two cases where C# would be best understood as a lower pitch than D♭. We should not make too much of this, for we have said that intonation was simply Riemann's marker for underscoring the functional differences between pitches. The more interesting aspect of Example 3-18 is Riemann's preference for 4b over 4a. At first glance the northwest reading looks quite good, because the triadic connections are all adjacent on the Table. In terms of pure intonation, 4a really is closer to chord 3 than 4b. On the other hand, Riemann's preference for the 3Q reading—for disjunct horizontal moves—suggests that distances on the Table do not always mirror perceived musical distances. This is a important charge, for if it is true we must reevaluate the Table as a model of tonal pitch space. It turns out that chord 4b is closer to chord 3 than first appears, and that the 4b reading faithfully reflects Riemann's sense of the relative strength of third and fifth relations: 'g' is preferred over 'e' as a common tone because it is the fifth of chord 3. To move from chord 3 to chord 4b, Riemann simply moves from 'c' to 'g', and reinterprets 'g' as the seventh of 'a'. In a sense, then, the modulation by 3Q is really a modulation by Q, and the relation between chords 3 and 4 is much closer than it appears. This relationship is obscured by the geometry of the Table, for there is no separate dimension for minor seventh relations; the seventh of a major-minor seventh will always lie 2Q away from the chord's root, as in the representations of chord 4 in Examples 3-16 and 3-17. In cases where a chordal seventh forges the link between two chords, the Table can not clearly depict the relationship. Are we to hear a 3Q relation between 'c' and 'a', and then count back 2Q for a net move of Q; or do we move 1Q directly, and infer the consonant a–$c^{\#}$–e from its dissonant seventh?

A few final words on the subject of intonation seem in order, particularly in connection with Riemann's tone *Verzeichniss*. The *Verzeichniss* should not be mistaken as an argument for just intonation. Scientists such as Oettingen and Helmholtz were far more serious advocates of pure tuning than musicians such as Riemann and Hauptmann. It sheds more light on Riemann's music psychology to consider the *Verzeichniss* as a set of "imaginary" counterparts to the fixed pitches of the equal-tempered scale. *Taste* C stands for all Cs (D^{bb}s, $B^{\#}$s, etc.); *Taste* $C^{\#}$ for all $C^{\#}$s (D^{b}s, $B^{\#\#}$s, etc.), and so on. Under this interpretation, the equal-tempered scale comprises a set of "dummy" pitches (although Riemann does not put it this way) out of which pure *Tonvorstellungen* are constructed. Perception acts upon the sensation of tempered pitch by "substituting" just-tempered values to fit the context.

We have seen this concept of substitution at work in Euler's theory, in connection with the seventh chord 36 : 45 : 45 : 64. It is also in keeping with Riemann's use of the Table (which outlasted his interest in just intonation), and in his view of perception as an active rather than passive faculty.

3.13 Musical Syntax

Riemann embarked upon a systematic treatment of chord connection in *Musikalische Syntaxis*: What was peripheral in the disseration now became the subject of an entire treatise. Though the Table did not appear in this work, Riemann's new focus on chord-to-chord relations set the tenor for its appearance in later works. It is worth noting that the *große Cadenz* was also absent, at least in its capacity as formal archetype. From the opening dedication onward, Riemann's inspiration was clearly the system of chord connection presented eleven years earlier in Oettingen's *Harmoniesystem*. But the departures from the latter, in respect to the phonic system and just intonation, were far from trivial. The "most essential point" (*wesentlichste Punkt*) in which Riemann's "principal representation (*Darstellung*) deviated from that of Oettingen" was his new embrace of enharmonic identification.[75] He maintained that the merits of pure tuning were far outweighed by the losses, which included having to judge classical masterpieces as swarming with "fehlerhaften Modulationen"—an obvious and disparaging reference to Oettingen's

[75] Riemann, *Musikalische Syntaxis*, vii.

earlier analysis of Beethoven's Piano Sonata in C major, op. 2, no. 3 (see 2.8).

Much of the introduction to *Musikalische Syntaxis* was in fact a defense of equal temperament, where Riemann drew up the boundaries between his views on intonation, consonance, and musical perception, and the views held jointly by Helmholtz and Oettingen (and most acousticians at the time). Riemann's defense confirms our earlier remarks about the tone *Verzeichniss* and the active role of human perception. Here Riemann maintained that pure tuning was unnecessary to our *Vorstellungsleben* because our interpretive faculties could compensate (*abfinden*) for tempered tones; he rejected the idea that consonance was solely a function of harmonic beating (*Schwebungen*), and that human perception was exclusively physiological. The term Riemann most often used in connection with Helmholtz's theory of perception was "Erleiden," which connoted passivity or sufferance, as opposed to active engagement. His own position stressed the comparative nature of perception, and the view that it was a logical rather than physical activity. We have mentioned these ideas before, but should add here that Riemann's views were intimately connected with his musical aesthetics, and greatly influenced by the philosophical-aesthetic writings of Hanslick, Fechner, Drobisch, and particularly Lotze.[76]

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[76] Riemann, *Musikalische Syntaxis*, x–xi, 1–4. Riemann cites G. T. Fechner's *Vorschule der Aesthetik*, 2 vols. (Leipzig: Breitkopf und Härtel, 1876), and Hanslick's *Vom Musikalisch-Schönen: ein Beitrag zur Revision der Ästhetik der Tonkunst* (Leipzig, 1854). There are further references to Drobisch's *Neue Darstellung der Logik nach ihrer einfachsten Verhältnisse, mit Rücksicht auf Mathematik und Naturwissenschaft*, 3d ed. (Leipzig: L. Voss, 1863), and to Lotze's *Geschichte der Ästhetik in Deutschland*, Geschichte der Wissenschaften in Deutschland: Neuere Zeit, no. 7 (Munich: J. G. Cotta, 1868).

Hermann Lotze was well known in German academic circles for his work in aesthetics and philosophy of mind. It had been Riemann's good fortune to secure Lotze and the musicologist Eduard Krüger for his Göttingen defense in 1873, even though he had not studied at Göttingen and had failed to get his dissertation accepted once before.[77] The most important idea that Riemann received from Lotze was the so-called *Trägheitsgesetz* or law of inertia, which he also referred to as the "Oekonomie des Vorstellens."[78] This law assumed that human perception was inclined toward the path of least resistance. Riemann claimed that a listener would naturally interpret the first chord in a piece as *Hauptklang*, and maintain this interpretation for as long as possible.[79] The law had clear implications for the Table, which Riemann did not fully articulate until "Ideen" (see 4.4). A related idea, not specific to Lotze, was that the apprehension of tonal connections was the chief source of pleasure in music (the simpler the connections, the better) and fundamental to aesthetic judgement. Both ideas fueled Riemann's approach in *Musikalische Syntaxis.*

The five chapters of *Musikalische Syntaxis* were in large part an anthology of chord progressions, and far removed in tone from the philosophical rumblings of the introduction. The work was organized rigorously around Riemann's concept of thesis, which no longer connoted the *große Cadenz* or Hauptmann's dialectical logic; a thesis was any

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[77] See Wolfgang Boetticher, "Eduard Krüger als Professor der Musikgeschichte an der Georgia Augusta," in *Musikwissenschaft und Musikpflege an der Georg-August-Universität Göttingen: Beiträge in ihrer Geschichte*, ed. Martin Staehelin (Göttingen: Vandenhoeck & Ruprecht, 1987), 78–89.
[78] Riemann, *Musikalische Syntaxis*, 15.
[79] Weber expounded a similar "Gesetz der Trägheit" in his *Versuch einer geordneten Theorie der Tonsetzkunst*, 3d rev. ed., vol. 2, 109. See also Janna K. Saslaw, "Gottfried Weber and Multiple Meaning," *Theoria* 5 (1990–91), 88.

harmonic progression through which a *Klang* was established as *Hauptklang*. The companion terms antithesis and synthesis cropped up occasionally but were marginal to Riemann's new definition. Instead of the functional terminology of "Musikalische Logik," Riemann introduced a new terminology, which emphasized the connection of theses and was taken partly from Oettingen. The terms Riemann borrowed were homonom and antinom (see 2.4), to which he added his own homolog and antilog. Whereas the former described the relative quality of chords, the latter described the intervallic direction between them: C major and G major were homonoms, but the move from C to G was homologic because G was part of the harmonic series of C (G to C was antilogic because C was below G and contrary to the direction of its harmonic series).[80] Riemann acknowledged that the new terminology was "ein wenig komplicirt," but argued that it was "clear and congruent" (*deckend*—in the sense of geometrical congruence) and brought to light the relational connection (*Verwandtschaftsbeziehung*) between chords.

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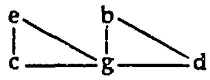
[80] Riemann, *Musikalische Syntaxis*, 15–16: "homonom (ομονομος): built according to the same principle; thus, a major chord if the other chord is major, a minor chord if the other is minor. Antinom is the opposite of homonom: If one chord is major, the other is minor. The terms homolog (ομολογος) and antilog (αντιλογος) actually indicate intervals rather than chords, albeit intervals in connection with a chord. An interval is homologic if it rises from the primary tone of a major chord (according to the principle that an ascent in the direction of the overtones is homologic in major), or falls from the primary tone of a minor chord (according to the principle that a descent in the direction of the undertones is homologic in minor). An antilogic interval is the opposite: a descent from the primary tone of a major chord, or ascent from the primary tone of a minor chord." [homonom (ομονομος): nach gleichem Princip aufgebaut, d. h. Dur, wenn der andere Klang Dur, Moll, wenn der andere Klang Moll ist. Das Gegentheil ist antinom, wenn der eine Klang Dur, der andere Moll ist. Homolog (ομολογος) oder antilog (αντιλογος) nennen wir eigentlich nicht Klänge, sondern Intervalle und zwar Intervalle in Beziehung auf einen Klang. Ein Intervall ist dann homolog, wenn es vom Haupttone eines Durakkordes ein steigendes (also dem Princip des Durgeschlechtes, das ein aufsteigen vom Haupttone nach der Obertonseite ist, entsprechendes, homologes), vom Haupttone eines Mollakkordes aber ein fallendes (also dem Princip des Mollgeschlechtes, das ein hinabsteigen vom Haupttone nach der Untertonseite ist, entsprechendes) ist. Antilog ist natürlich das Gegentheil, vom Durakkorde aus ein fallendes, vom Mollakkorde ein steigendes Intervall.]

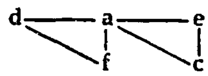
The simplest thesis of all was an unaccompanied tonic triad, as at the beginning of Beethoven's Piano Sonata in F minor, op. 57 (Riemann felt this music to be a strong affirmation of harmonic undertones as well). The next simplest variety was "one-sided thesis through two chords," the subject of Riemann's first chapter. "One-sided" progressions consisted entirely of homologic or antilogic chords relative to the tonic. Progressions involving chords on both sides of the tonic (IV–V–I) were "two-sided." Riemann gave twenty examples of one-sided thesis with two chords. These were organized into three groups: A) thesis through homologic-homonomic *Klänge;* B) thesis through antilogic-homonomic *Klänge;* and C) thesis through antinomic *Klänge.* Riemann labeled the progressions using a new *Klangschlüssel* notation that he derived from Oettingen's *Harmoniesystem.* Example 3-19 maps all three groups of "one-sided thesis through two chords" onto the Table, and gives the *Klangschlüssel* and pertinent terminology for each progression. We shall refer back to this example in Chapter 5.

EXAMPLE 3-19: ONE-SIDED THESIS THROUGH TWO CHORDS

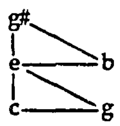
A: Thesis through homologic-homonomic Klänge

1) 

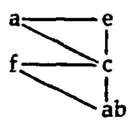
Klangschlüssel: c+ – g+ – c+
Chord Names: +Tonika, +Oberdominante
Relationship: homologic-homonomicQuintklang

2) 

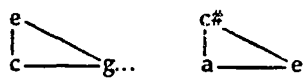
Klangschlüssel: °e – °a – °e
Chord Names: °Tonika, °Unterdominante
Relationship: homologic-homonomicQuintklang

3) 

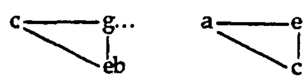
Klangschlüssel: c+ – e+ – c+
Chord Names: +Tonika, +Terzklang
Relationship: homologic-homonomicTerzklang

4) 

Klangschlüssel: °e – °c – °e
Chord Names: °Tonika, °Terzklang
Relationship: homologic-homonomicTerzklang

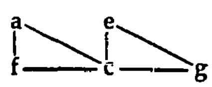
5) 

Klangschlüssel: c+ – a+ – c+
Chord Names: +Tonika, +Sextenklang
Relationship: homologic-homonomicSextenklang

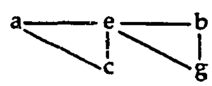
6) 

Klangschlüssel: °e – °g – °e
Chord Names: °Tonika, °Unterdominante
Relationship: homologic-homonomicQuintklang

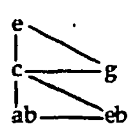
B: Thesis through antilogic-homonomic Klänge

7) 

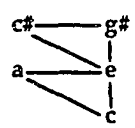
Klangschlüssel: c+ – f+ – c+
Chord Names: +Tonika, +Unterdominante
Relationship: antilogic-homonomicQuintklang

8) 

Klangschlüssel: °e – °b – °e
Chord Names: °Tonika, °Oberdominante
Relationship: antilogic-homonomicQuintklang

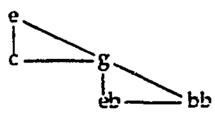
9) 

Klangschlüssel: c+ – ab+ – c+
Chord Names: +Tonika, +Terzklang
Relationship: antilogic-homonomicTerzklang

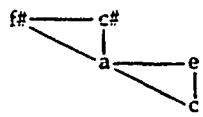
10) 

Klangschlüssel: °e – °g# – °e
Chord Names: °Tonika, °Terzklang
Relationship: antilogic-homonomicTerzklang

EXAMPLE 3-19 (CONT): ONE-SIDED THESIS THROUGH TWO CHORDS

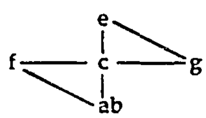
11) 

Klangschlüssel: c+ – eb+ – c+
Chord Names: +Tonika, +Sextenklang
Relationship: antilogic-homonomic Sextenklang

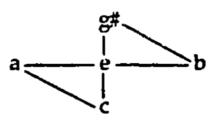
12) 

Klangschlüssel: °e – °c# – °e
Chord Names: °Tonika, °Sextenklang
Relationship: antilogic-homonomic Sextenklang

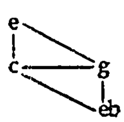
C: Thesis through antinomic Klänge

13) 

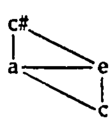
Klangschlüssel: c+ – °c – c+
Chord Names: +Tonika, °Unterdominante
Relationship: antinomic Wechsel

14) 

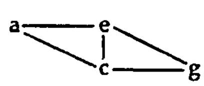
Klangschlüssel: °e – e+ – °e
Chord Names: °Tonika, +Oberdominante
Relationship: antinomic Wechsel

15) 

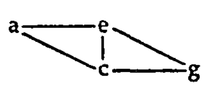
Klangschlüssel: c+ – °g – c+
Chord Names: +Tonika, °Quintwechselklang
Relationship: homologic-antinomic Quintklang

16) 

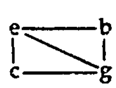
Klangschlüssel: °e – +a – °e
Chord Names: °Tonika, +Quintwechselklang
Relationship: homologic-antinomic Quintklang

17) 

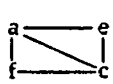
Klangschlüssel: +c – °e – +c
Chord Names: +Tonika, °Terzwechselklang
Relationship: homologic-antinomic Terzklang

18) 

Klangschlüssel: °e – +c – °e
Chord Names: °Tonika, +Terzwechselklang
Relationship: homologic-antinomic Terzklang

19) 

Klangschlüssel: +c – °b – +c
Chord Names: +Tonika, °Sextenwechselklang
Relationship: homologic-antinomic Sextenklang

20) 

Klangschlüssel: °e – f+ – °e
Chord Names: °Tonika, +Sextenwechselklang
Relationship: homologic-antinomic Sextenklang

Riemann likened the progressions of Example 3-19 to simple linguistic formations; the tonic was analogous to a subject, its confirmation was analogous to a predicate.[81] The four progressions involving homonomic *Sextenklänge* (nos. 5, 6, 11, and 12) were more obscure than the others because the sixth was an indirectly comprehensible interval. Numbers 5 and 6 could possibly be construed as two-sided theses around G major and G minor, respectively. Though all four progressions were common, Riemann said that the sixth-step was used to better effect in three-chord theses. The two progressions involving antinomic *Sextenklänge* (nos. 19, 20) were viable as two-chord theses; Riemann's terminology, however, is in need of clarification. He called the chord e–g–b in no. 19 a *Sextenwechselklang* because the chord was derived from c–e–g by changing the meaning of the major sixth g–e: With 'c' as tonic, the interval g–e is expressed by the ratio 3 : 5; with 'b' as phonic, the interval e–g is expressed by the ratio $^1/_3 : {}^1/_5$. Thus, in the progression from c–e–g to e–g–b, the meaning of the interval e–g is inverted from 3 : 5 to $^1/_3 : {}^1/_5$. *Sextenwechselklänge* were later renamed *Leittonwechselklänge.*

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[81] Riemann, *Musikalische Syntaxis,* 23. Riemann writes: "If one wanted an analogy, one could perhaps compare simple theses to the formation of the basic components of a sentence. The subject would be the tonic, and the predicate would be its confirmation through the *Quint-, Terz-,* or some other *Klang.*" [Wollte man durchaus Analogien, so könnte man die obigen einfachen Thesen etwa der sprachlichen Satzbildungsform des einfachen Urtheils vergleichen. Das Subjekt würde die Tonika sein und das Prädikat die Bestätigung durch den Quint-, Terz- u. Klang.]

3.14 Syntax in Schubert's Impromptu op. 90, no. 3

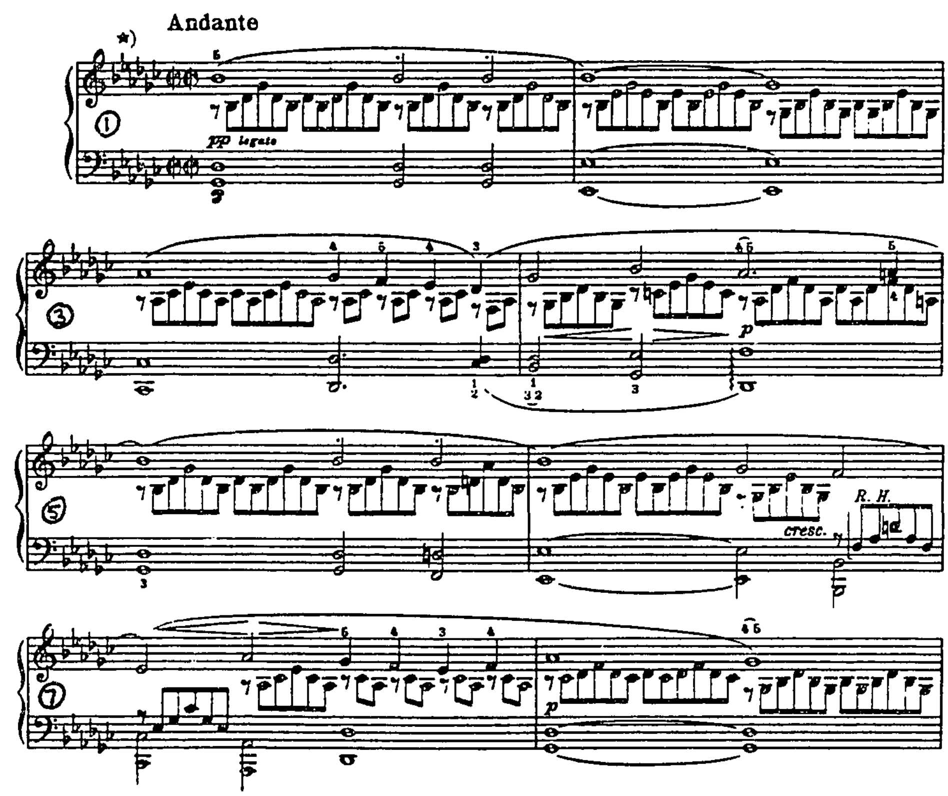
In subsequent chapters Riemann treated one-sided thesis through three chords, two-sided thesis, and, finally, the conjunction of theses (*Thesenverkettung*), at which stage he was ready to apply his results to actual music. Chapter 5 contained a detailed analysis of Schubert's Impromptu in G^b major, op. 90, no. 3, which pointed up the differences between musical logic as seen through the *große Cadenz*, and Riemann's new musical syntax. Example 3-20 summarizes Riemann's analysis of the first two phrases.[82] We have provided roman numerals beneath the *Klangschlüssel* for ease of reference.

[82] Riemann, *Musikalische Syntaxis*, 66. Riemann used the G major version of this work, and rebarred it so that two of his measures corresponded to one of Schubert's. We preserve Schubert's barring and transpose Riemann's analysis from G major to G^b major. Riemann does not give *Klangschlüssel* for the consequent phrase (mm. 5–8 in our example), but writes the following: "In a second 8-measure (direct and closed) thesis, Schubert actually closes on g+, which remains tonic throughout, so that both 8-measure phrases belong together and act as antecedent and consequent." [In einer zweiten 8 taktigen (direkten und geschlossenen) These schließt nun Schubert wirklich auf g+, das durchweg Tonika bleibt, so dass die beiden 8 taktigen Sätzchen zusammengehören und sich wie Vorder- und Nachsatz verhalten.] Riemann's use of *Vordersatz* and *Nachsatz* is appropriate to his new syntactic outlook since these are the terms, respectively, for the protasis and apodosis of a conditional sentence.

EXAMPLE 3-20: IMPROMPTU IN G-FLAT MAJOR, OP. 90, NO. 3. MM. 1–8

<u>Phrase 1</u> (mm. 1–4):	1	2	3	4
Klangschlüssel:	$g^{b}+$	$°b^{b}$	$°e^{b}$ $d^{b}+^{7}$	$g^{b}+$ $_{7}°b^{b}$ $d^{b}+$
Roman numerals:	I	vi	ii^{6} V^{7}	I^{6} vii$^{o4}_{3}$ V
Thesis:	$g^{b}+$: direct-open, one-sided			

<u>Phrase 2</u> (mm. 5–8):	5	6	7	8
Klangschlüssel:	$g^{b}+$	$°b^{b}$ $b^{b}+^{7}$	$c^{b}+$ $°e^{b}$ $d^{b}+^{7}$	$g^{b}+$
Roman numerals:	I	vi V7 (vi)	IV ii V7	I
Thesis:	$g^{b}+$: direct-closed, two-sided			



The antecedent phrase of mm. 1–4 was a "direct-open" (*direkt, offen*) thesis, because it began on the tonic and ended with a half cadence. An "indirect-closed" (*mittelbar, geschlossen*) thesis would close on the tonic and begin on some nontonic chord. The antecedent was also a one-sided thesis, although this was less clear. In Riemann's view, the fact that $°b^b$ and $°e^b$ occurred in both G^b major and C^b major gave measures 1–4 a flavor (*Beigeschmack*) of antithesis or two-sidedness. The consequent phrase of mm. 5–8 was "direct-closed" (*direkt, geschlossen*) and two-sided on account of the antilogic-homonomic *Quintklang* c^b+ and its offshoot $°e^b$.

The next two theses occurred in measures 9–12 (phrase 3) and were indirect-closed in $°e^b$ and c^b+, respectively. According to Riemann, these theses were too transient to have the effect of a modulation and instead formed a higher-level two-sided thesis with the music of mm. 1–8. The *Beigeschmack* of mm. 2–3 was thus intensified. An indirect-closed thesis in g^b+ followed in mm. 13–16 (phrase 4). Riemann analysed this phrase into smaller thetic fragments, to show that the concept of thesis was operative at different levels. Example 3-21 summarizes Riemann's analysis of mm. 9–16.

EXAMPLE 3-21: IMPROMPTU IN G-FLAT MAJOR, OP. 90, NO. 3. MM. 9–16

Phrase 3 (mm. 9–12):	9	10	11	12
Klangschlüssel:	$d^{b}+$ $e^{b}+^{7}$	$^{\circ}e^{b}$	$f^{b}+$ $c^{b}+$ $g^{b}+^{7}$	$c^{b}+$
Roman numerals:	IV V	i	IV I^{6} V	I
Thesis:	$^{\circ}e^{b}$: indirect-closed; two-sided		$c^{b}+$: indirect-closed two-sided	

Phrase 4 (mm. 13–16):	13	14	15	16
Klangschlüssel:	$c^{b}+$ $d^{b}+$	$g^{b}+$ $c^{b}+$ $^{\circ}e^{b}$	$g^{b}+$ $d^{b}+^{7}$	$g^{b}+$
Roman numerals:	IV V	I^{6} IV ii^{6}	I V	I
Thesis:	$g^{b}+$: indirect-closed, two-sided			
Thetic Fragments:	$d^{b}+$: indirect-open, two-sided (m. 13) $g^{b}+$: direct-open, two-sided (m. 14) $g^{b}+$: direct-closed, one-sided (mm. 15–16)[83]			

[83] Riemann, *Musikalische Syntaxis*, 67. Riemann's analysis of this last thetic fragment as direct suggests that he viewed the cadential six-four as being tonic rather than dominant in function. Compare with Ex. 3-3 above, however, where the cadential six-four is grouped with subdominant harmony as part of the *Antithese*. The usual dominant reading of this chord is the one interpretation that Riemann did not make.

Riemann went on in similar fashion for the contrasting middle section and recapitulation. In the coda, he noted the striking move from G^b major to the minor Neapolitan (°e^{bb}, enharmonically °d), which he termed the antilogic-antinomic *Terztonart*. He then made a few summary comments concerning the grouping of theses within the whole, and the connection of individual theses. He distinguished in particular the "harmonic step" leading from the end of one thesis to the beginning of another, from the "system step" between the tonics of the two theses. Riemann clarified the distinction with the following theses in c+ and f+: f+ – $g+^7$ – f+ – $g+^7$, and b^b+ – $c+^7$ – b^b+ – $c+^7$.[84] Both are indirect-open and two-sided, which means that neither actually states the tonic. This is the sort of tonic-by-circumscription that we noted in connection with Example 3-15. The harmonic step consists in the linking move $g+^7$ – b^b, whereas the system step consists of the move c+ – f+. The concept of system step led Riemann into a discussion of key relations, using the general approach of Example 3-21 but now applied to a higher level of musical structure. System steps were ultimately more important than harmonic steps because they clarified obscure connections within theses. In the theses $b+^7$ – c+ and °c – $b+^7$, for example, knowing that c+ – °c is an antinomic *Wechsel* does not help one to understand the relation °c – $b+^7$.[85] Knowing that the system step is c+ – e+, however, reveals °c as the antilogic-antinomic *Terzklang* (the minor Neapolitan) and $b+^7$ as the *Dominante*. Riemann used the same terms to describe system steps as to

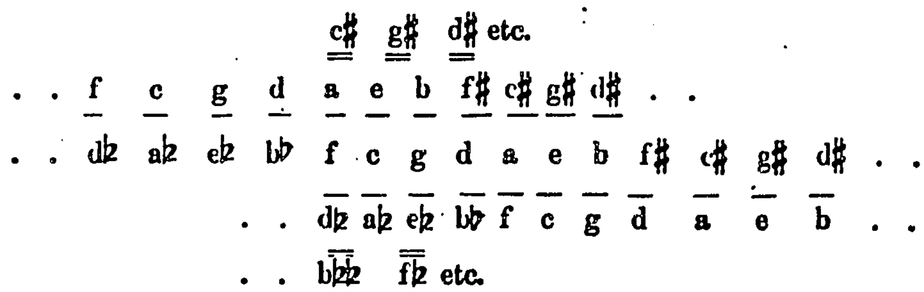
[84] Riemann, *Musikalische Syntaxis*, 85.
[85] Riemann, *Musikalische Syntaxis*, 85. Riemann discusses the theses b+7 – c+ and °c – b+7 in connection with one of his own compositions, a Waltz in A^b minor for which he provides no example.

describe chord connections; thus, the mappings in Example 3-20 could apply as well to *Tonarten* as to *Klänge.*

3.15 "Die Natur der Harmonik"

We are now in a position to make some general distinctions between musical logic and musical syntax. Before doing so, however, we should say a few words about "Die Natur der Harmonik," an uncommonly readable work that was in some respects a précis of the first decade of Riemann's career. This essay—just 32 pages in the English translation—was probably aimed at a less specialized readership than Riemann's earlier writings. Riemann sketched the history of harmonic dualism and outlined the chief contributions of theorists from Rameau to Oettingen. The essay gave a clear sense of where Riemann stood on various issues after ten years of speculation. Toward the end, the Table appeared in a brief but interesting discussion of third relations. We reproduce it below in Example 3-21.

EXAMPLE 3-22: THE TABLE OF RELATIONS FROM "DIE NATUR DER HARMONIK."[86]



In outward appearance this version of the Table was not much different from the version presented in the dissertation. Riemann retained the *Striche* with the same meaning as before, but now used lowercase *Buchstaben* exclusively. Of the horizontal and vertical series, however, Riemann claimed that the most important aspect was "the recognition of third-relationship in tones and in keys." Third relations had earlier been deemed subordinate to fifth relations but Riemann now believed they were "equally valid with fifth-relationships." Traditionally, according to Riemann, theorists had wondered why keys such as D major or B^{b} major sounded remote from C major, whereas E major and A^{b} major—which were more remote on the circle-of-fifths—sounded relatively close. The reason, made plain by the Table of Relations, was that "E major is not the key of the fourth fifth, but the key of the third."[87]

[86] Riemann, "Die Natur der Harmonik," *Sammlung musikalischer Vorträge: Ein Almanach für die musikalische Welt* 4/40 (1882): 157–90; trans. J. F. Comfort as "The Nature of Harmony," in *New Lessons in Harmony* (Philadelphia: Presser, 1886), 27.
[87] Riemann, "The Nature of Harmony," 26.

Riemann mentioned the second subject of Beethoven's Piano Sonata in C major, op. 53 as an example of direct key relationship by third.[88]

Riemann had revised his opinions on other fronts as well. Gone, for example, was the Oettingen-inspired terminology of *Musikalische Syntaxis*, which Riemann had adopted for its clarity and congruence. This change had actually occurred a couple of years earlier, in the first edition of *Skizze einer neuen Methode der Harmonielehre* (1880), where homologic and antilogic were replaced by *Schritt* and *Gegenschritt*, and antinomic relations became *Wechsel* relations; F major was no longer the antilogic-homonomic *Quintklang* of C major, but the *Gegenquintklang* that resulted through a *Gegenquintschritt*. We shall discuss this new terminology and its implications in Chapter 5.

3.16 Summary

We began this chapter by suggesting two opposing notions of harmonic function, the categorical and the chordal. The first of these maintained that chord and function were distinct, whereas the second blurred the distinction by anchoring specific functions to specific chords. Riemann's model of categorical function was the *große Cadenz*, whose three functional stages—thesis, antithesis, and synthesis—were believed to be manifest at all levels of musical structure. The dialectical relationship between these stages was the core of Riemann's notion of musical logic.

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[88] Orthodox Schenkerians view this E major as a "way-station" on route to the dominant. See Oswald Jonas, *An Introduction to the Theory of Heinrich Schenker: The Nature of the Musical Work of Art* (Longman: New York, 1982), 35.

The association of musical logic, the *große Cadenz*, and categorical function is found in Riemann's earliest work, where the influence of Hauptmann is strongest.

Chordal function treats the manner of connection between chords—whether a chord is antilogic/antinomic, homologic/homonomic relative to another chord—rather than the broader logic governing chord progressions. Since chord and function merge under this view, it no longer makes sense to speak of a functional dialectic manifest at different levels of musical structure. Riemann's shift away from musical logic, toward the connective or syntactic aspect of harmony is accompanied by a shift of paradigms; the *große Cadenz* gives way to the Table of Relations as a model of harmonic function. The association of musical syntax, the Table of Relations, and chordal function comes increasingly to the fore after "Ueber das musikalische Hören," and coincides with Riemann's exposure to the Oettingen's theory of harmonic dualism.

Riemann's concepts of musical logic and musical syntax entailed very different approaches to the study of harmony. A few of the most important differences are listed below:

1) Musical logic interpreted chords dialectically, as belonging to the thetic, antithetic, or synthetic stage of the *große Cadenz*. One and the same chord might participate at each stage but retain a distinct meaning by virtue of its position in the cadence. This positional or contextual aspect of musical logic is its most salient feature.

Musical syntax described chord-to-chord relations on the basis of Oettingen's theory of chord connection, but in isolation from any larger context.

2) Musical logic was hierarchical since its paradigm—the *große Cadenz*—was manifest at all levels of musical structure. Riemann's early analyses demonstrate this hierarchical aspect; the *große Cadenz* is implicit in local melodic successions, as well as over larger harmonic spans.

Musical syntax tended toward the linkage of harmonic theses, rather than the embedding of cadential functions.

3) Musical logic was abstract but systematic; chords stood for functional categories whose relationship was consistent from one level of musical structure to another.

Musical syntax was concrete, in the sense that chords were no longer agents of abstract functions. On the other hand, the "logic" of chordal connection was hard to pin down since musical syntax provided no context for assessing the behavior of chords.

4) Musical logic implied a parsing of musical structure along the lines of the *große Cadenz*. Musical structures were thus approached as synchronic wholes.

Musical syntax implied a more dynamic concatentation of harmonic theses—a *Thesenverkettung*—and encouraged a diachronic view of musical structure.

5) Musical logic used various modes of *Erweiterung* to elaborate the *große Cadenz*.

Musical syntax used homologic/antilogic-homonomic/antinomic transformations to construct harmonic theses.

Chapter 4: Helmholtz, Riemann, and the Idea of Tonal Representation

4.1 Introduction

In this chapter we shall compare the approaches to music perception taken by Helmholtz (1821–94) in his *Lehre von den Tonempfindungen,*[1] and some fifty years later by Riemann (1849–1919) in the essay "Ideen zu einer 'Lehre von den Tonvorstellungen.'"[2] The differences between Helmholtz and Riemann, hinted at in the titles of these works, were considerable and crucial for the development of music psychology. Since most of what follows concerns these differences, we begin by suggesting a line of continuity between the two men.

Although Helmholtz and Riemann approached the problem of perception from contrary points of view, we know that Riemann believed he was forwarding a tradition to which the older scientist's music-theoretical work also belonged. The treatment of the physiological and acoustical aspects of hearing in "Ueber das musikalische Hören" came directly from *Tonempfindungen,* as did most of Riemann's technical knowledge of these subjects in later works. If there was tension, it was not between Riemann and Helmholtz per se, but consisted in Helmholtz's renunciation of idealist philosophy—the wellspring of much nineteenth-century scientific thought in Germany—for a stringent empiricism that

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[1] We use the Ellis translation of the 4th German edition (1877) throughout this chapter (see Chap. 2 n. 19). German citations are from the 6th ed. (1913; reprint, Hildesheim: Georg Olms, 1968), which corrects printing errors found in earlier editions but otherwise is the same as the 4th edition.

[2] See Chap. 2 n. 21. Riemann's essay has been translated by Robert Wason and Elizabeth West Marvin as "Riemann's 'Ideen zu einer Lehre von den Tonvorstellungen': An Annotated Translation," *Journal of Music Theory* 36/1 (1992): 69–117.

accepted nothing on faith. Helmholtz's empiricism would be unremarkable but for the fact that the fruits of his method drew public and scholarly acclaim at a time of vigorous neo-Kantian revival.[3] We shall return to German idealism below; it suffices to say here that Helmholtz's empiricism enjoyed ever-growing prestige and influence in the last decades of the nineteenth century.

Riemann was not unmoved by the flow of science toward the veracities of human experience, and spent the better part of his career developing a music theory that set forth the "logical activity" of hearing. His differences with Helmholtz were substantial, but, with the exception of the undertone debacle, did not impinge upon the empirical integrity of Helmholtz's research. Riemann recognized this research as pathbreaking and acknowledged its authority on several occasions. He believed, however, that psychoacoustics had limited jurisdiction over music-psychological behavior; the mass of data that formed the core of Helmholtz's music theory was thus peripheral to Riemann, and psychoacoustical facts—if they were consulted at all—remained subordinate to higher-level conjectures about music cognition.

Our treatment of Helmholtz and Riemann falls into three parts. First we discuss Helmholtz in historical terms, treating both his empiricism and his notion of mental representation, or *Vorstellung*. Next we explore Helmholtz's contributions to music theory, concentrating on the idea of tonal motion, which was central to his theory of music perception.

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[3] Much of the revived interest in Kant was centered in Marburg. In 1865 Otto Liebmann published a paper whose spirited conclusion, "es muss auf Kant zurückgegangen werden" became a slogan for the "back to Kant" movement. Several German intellectuals, including Hermann Cohen, his student Ernst Cassirer, and the sociologist Georg Simmel embraced and contributed to the renewed interest in Kant. For the relation between the neo-Kantians and emerging modernism see Peter Gay, "Encounter with Modernism" in *Freud, Jews and Other Germans* (New York: Oxford Univ. Press, 1978), 93–168.

Related to tonal motion is tonal distance: If we say that we "move" from one tone, chord, or key to another, it makes sense to ask where and how far we have gone.

The chief differences between Helmholtz and Riemann underscore a distinction made at the turn of the century between *Tonpsychologie* and *Musikpsychologie*.[4] Helmholtz was a "tone psychologist" insofar as his conclusions about music perception rested on data gathered in isolation from musical contexts. Riemann's pursuit of encompassing perceptual laws—laws that guided listeners through actual pieces of music—was music psychological. In the third part of this chapter, we compare Riemann's and Helmholtz's respective formulations of tonal distance and tonal representation. Both referred to *Vorstellungen* in their music-theoretical writings, but their attitudes toward this concept differed considerably. By distinguishing these attitudes we may begin to retrace the path leading from the early tone-psychological investigations of Helmholtz to Riemann's pioneering efforts in music psychology.

4.2 Helmholtz in Historical Context

The surge of interest in music perception in the late nineteenth century was not so much an overture as a denouement to a drama that began several decades earlier. A loss of confidence in prevailing theories of knowledge lay at the heart of this drama, and a struggle ensued between upholders of the view that knowledge accorded with certain universal principles, and a younger generation who assailed this view as

[4] For a useful overview of nineteenth-century theories of music perception, see Elizabeth West Marvin, "*Tonpsychologie* and *Musikpsychologie*. Historical Perspectives on the Study of Music Perception." *Theoria* 2 (1987): 59–84.

superstitious and untenable. The eminent German physicist and physiologist Hermann von Helmholtz was among this younger generation. Helmholtz rejected the notion of a priori knowledge, and insisted that all knowledge came directly through the senses. His classic study in physiological acoustics *Die Lehre von den Tonempfindungen* approached music theory freshly by asking how people hear.

Even by nineteenth-century standards, Helmholtz's work displayed an incredible range. Besides acoustics, Helmholtz made significant contributions to the fields of physiology, optics, thermodynamics, and meteorology. As a young man he had wanted to become a physicist, but for financial reasons ended up accepting a medical scholarship and studying in Berlin with the famous physiologist Johannes Müller. In 1847, at the age of twenty-six, Helmholtz published a paper on the conservation of energy that was epochal in the fields of physiology and physics.[5] A few years later he invented the indirect ophthalmoscope, an instrument used to examine retinae and determine the proper strength of prescription lenses. This invention brought Helmholtz considerable fame and may have prompted his further investigations into sensation and perception; the results of these investigations became known with the publication in

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[5] Helmholtz, *Über die Erhaltung der Kraft* (Berlin: G. Reimer, 1847); reprinted in *Wissenschaftliche Abhandlungen von Hermann Helmholtz*, vol. 1 (Leipzig: J. A. Barth, 1882), 12–75. Helmholtz proved that "vital" heat—heat that was not the result of chemical reactions within an organism—could be explained by physical forces within the organism. Vitalists such as Johannes Müller believed that vital heat was one aspect of an inscrutable life force that coordinated the physiological functions of living creatures. Helmholtz did not accept the notion of a metaphysical life force beyond the scope of scientific inquiry. He argued that all heat was reducible to ordinary mechanical forces, and that force itself was not created or destroyed but simply tranformed from one state to another. By adopting mechanism in place of vitalism, Helmholtz was declaring that all physiological phenomena were susceptible to analysis. This spirit of inquiry fueled his later researches into visual and auditory perception. See *Encylopaedia Britannica*, 15th ed., s.v. "Helmholtz," and Raymond E. Fancher, *Pioneers of Psychology* (New York: Norton, 1979), 97.

1856 of the first volume of his *Handbuch der physiologischen Optik.* Two remaining volumes brought this work to completion in 1867, but not before Helmholtz had published the first edition of his masterly *Lehre von den Tonempfindungen.*

One thread that runs through Helmholtz's work is the rejection of *Naturphilosophie,* a movement that opposed Newtonian science and promoted an epistemology based on subjective experience and intuition. Helmholtz's attitude toward this movement is best glimpsed in connection with its leading figure, the German poet-dramatist Johann Wolfgang von Goethe (1749–1832). Apart from a remarkable literary output, Goethe wrote extensively and knowledgeably on a wide range of scientific topics. Helmholtz believed that the poet's "Erster Entwurf einer allgemeinen Einleitung in die vergleichende Anatomie" (1795)[6] made substantial contributions to comparative anatomy, but that Goethe was less successful when he turned his attention to the study of light and the nature of vision.

Helmholtz was thinking specifically of Goethe's *Farbenlehre* (1810), which had challenged the basic principles of Newtonian optics. Newton had shown in the famous "experimentum crucis" that white light was compound rather than primary because it broke up into a spectrum of seven colors when refracted through a prism. Goethe denied that white light, the source of our purest visual sensations, was compounded of other

[6] Johann Wolfgang von Goethe, "Erster Entwurf einer allgemeinen Einleitung in die vergleichende Anatomie, ausgehend von der Osteologie," vol. 17 of *Gedenkausgabe der Werke, Briefe und Gespräche,* 2d ed. (Zürich: Artemis Verlags, 1966), 231–69. Goethe asserted in this work, some sixty-five years before Darwin, that all vertebrates were variations on a primitive morphological theme. Our synopsis of Helmholtz is based on Richard M. Warren, "Helmholtz and His Continuing Influence," *Music Perception* 1/3 (1984): 253–55, and Richard M. Warren and Rosyln P. Warren, *Helmholtz on Perception: Its Physiology and Development* (New York: Wiley, 1968), 1–23.

colors: "The claim that all the colors form white when they are mixed together is an absurdity which, along with other absurdities, people have become used to repeating faithfully and contrary to appearances for a century now."[7] He conjectured that the prism added colors to the white light and thus corrupted it. In place of Newton's prism, Goethe appealed to the power of the senses. If white light really were impure, he argued, we would know this without the intervention of a prism. Helmholtz knew otherwise; in comparing the sensation of sound to that of color, he wrote that the "power of distinguishing the different elementary constituents of the sensation is originally absent in the sense of sight."[8] He attributed this impotence to the lack of an "exact and certain image in our recollection [*Erinnerungsbild*]" of the colors of the spectrum. Our experience is limited instead to pigments that approximate these colors and behave differently from them. A mixture of blue and yellow pigments will produce green, whereas one of blue and yellow beams of light will produce white. It was significant to Helmholtz that earlier theories of color were based on pigments rather than colored light: "To this very circumstance is due the violent opposition of Goethe, who was only acquainted with the colors of pigments, to the assertion that white was a mixture of variously colored beams of light."[9] Goethe's ignorance of physics did not invalidate his *Farbenlehre*; his faith in the senses and

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[7] Johann Wolfgang von Goethe, *Zur Farbenlehre. Das Gesamte Hauptwerk von 1810*, vol. 23/1 of *Sämtliche Werke: Briefe, Tagebücher und Gespräche* (Frankfurt am Main: Deutscher Klassiker Verlag, 1991), 189. [Daß alle Farben zusammengemischt Weiß machen, ist eine Absurdität, die man nebst andern Absurditäten schon ein Jahrhundert gläubig und dem Augenschein entgegen zu wiederholen gewohnt ist.]

[8] Helmholtz, 111; Ellis trans., 64. [dem Gesichtssinn von vornherein die Fähigkeit, die verschiedenen elementaren Bestandteile der Empfindung zu scheiden, fehlt]

[9] Helmholtz, 111; Ellis trans., 64. [Eben dahin gehört die heftige Opposition Goethes, der auch nur die Mischung der Pigmentfarben kannte, gegen den Satz, daß Weiß eine Mischung verschiedenfarbigen Lichtes sein könne.]

mistrust of abstraction clearly were strengths, not just weaknesses, and the work held its own as a carefully argued theory of perception (it was admired by the likes of Schelling, Hegel, and Schopenhauer). But the triumph of subjectivity we find here and elsewhere in Goethe had little in common with the empiricism espoused by Helmholtz. Scientific inquiry, Helmholtz maintained, often involved the examination of phenomena that were hidden from the senses, such as the spectral components of white light or the higher partials of sound. Our sensations, accordingly, give no exact picture of events in the physical world, but are merely signs that coalesce into mental representations or *Vorstellungen* of these events.

There are two similar words in German for representation: *Vorstellung* and *Darstellung*. The former is a mental picture—"idea" or "conception" are alternative translations—whereas the latter is an objective or schematic representation suggesting nothing of the state of mind of the subject. A painting might convey the idea (*Vorstellung*) of a cathedral without depicting it in a striking or even discernable fashion: "Das Bild stellt Notre Dame vor." An architectural blueprint (*Darstellung*) would evoke the same edifice, but with a wealth of objective detail: "Das Bild stellt Notre Dame dar." *Vorstellungen* concern mental images—their clarity, intensity, persistence—and *Darstellungen* concern external reality. It is improper to use the terms synonymously, and Helmholtz consistently observes their distinction by restricting *Vorstellung* to perceptual phenomena. Such representations are psychological, though obviously conditioned by sensory (i.e. physiological) input.

Helmholtz divides *Vorstellungen* into two broad classes: those whose sensory components can be isolated in perception, and those which admit of no such analysis. The latter belong to a "lower grade of consciousness" ("niedere Grad des Bewußtwerdens"), since the underlying sense data make only a collective impression on perception. A "second and higher grade" pertains to *Vorstellungen* whose sensations are open to analytic introspection. The terms "perzipiert" and "apperzipiert" describe, respectively, the synthetic nature of the lower grade and the analytic nature of the higher grade;[10] while there is much in music that permits and even depends upon analytic perception, Helmholtz believes (along with his predecessors Seebeck and Ohm) that in the case of partial tones, "the ease and exactness of the analytical perception [Apperzeption] is far behind that of the synthetic perception [Perzeption]."[11]

Whether *Vorstellungen* belong to a higher or lower grade is largely irrelevant to the formation of *Darstellungen,* since *Darstellungen* model objects of perception rather than perceptions themselves. In music these objects run the gamut of complexity from pitches and chords to full-fledged systems such as tonality. To model pitch, Helmholtz uses wave-form diagrams whose curves represent (*darstellen*) the physical motions that produce partial tones. Whatever these diagrams assert about the structure of pitch, they do so independently of particular *Vorstellungen,* for they are indifferent to whether partial tones are "perzipiert" or "apperzipiert." The gap between *Vorstellung* and *Darstellung* remains wide even when perception is most acute: Under favorable conditions one might perceive harmonics up to the seventh degree; even so, one

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[10] Helmholtz, 107; Ellis trans., 62. We adopt Ellis's usage of "synthetic" and "analytic" perception for the terms "perzipiert" and "apperziert." Helmholtz acknowledges that his own usage stems from Leibniz, although unlike Leibniz he sometimes uses the commonplace "wahrgenommen" interchangeably with "apperziert."

[11] Helmholtz, 109; Ellis trans., 63. [die Leichtigkeit und Genauigkeit der Apperzeption hinter der der Perzeption weit zurück.]

would not apprehend the wave forms or fluctuations in air pressure associated with these tones.

What then is the relation between *Vorstellungen* and *Darstellungen*? A blueprint is not a mental image and though the two may share features, it seems that one can exist quite separately from the other. Modern physics has proceeded as if this were the case; still, the distinction between *Vorstellung* and *Darstellung* begins to dissolve when we ask how objects are known in the first place. Epistemology forces a reconciliation of perception and the objects of perception, of the personal and impersonal: "It is only when we ask how physics can be *known*," writes Bertrand Russell, "that the importance of sense-data re-emerges."[12] It will serve our purposes to say that *Vorstellungen* and *Darstellungen* are nominally related but essentially different forms of representation. The former arise spontaneously from sense data, the latter through a process of conceptualization. *Darstellungen* may be inferred from *Vorstellungen*, and assist in the process of understanding them, but to the extent that we distinguish objects from perceptions, *Darstellungen* model the former and not the latter.[13]

The two concepts of representation played an important role in many epistemological queries in the late nineteenth century. In a public address given at the University of Berlin in 1878, Helmholtz asserted that "the basic problem the age placed at the beginning of all science was one of epistemology: 'What is truth in our perception and our thinking? In

12 Bertrand Russell, "The Relation of Sense-Data to Physics" in *Philosophy of Science*, ed. Arthur Danto and Sidney Morgenbesser (New York: Meridian, 1974), 36.
13 For a wider survey of the *Vorstellung-Darstellung* distinction, see Stephen Toulmin and Allan Janik, "Language, Ethics and Representation," in *Wittgenstein's Vienna* (New York: Simon and Schuster, 1973), 120–66.

what sense do our ideas [*Vorstellungen*] correspond to reality?'"[14] Is knowledge primarily a function of *Vorstellungen,* or is it conditioned (and to what extent) by *Darstellungen*?[15] This problem was compounded by nineteenth-century writers who failed to use the terms consistently, to distinguish direct, intuitive, and private knowledge on one hand, from speculative, discursive, and public knowledge on the other.[16] Neither Helmholtz nor Riemann mixed up their terms, but Riemann's epistemology was tied to a particular *Darstellung*—the Table of Relations—whereas Helmholtz's drew its nourishment from *Vorstellungen,* a difference due mainly to the differing agendas of tone psychology and music psychology. Helmholtz would have had no problem with the definition of *Vorstellung* set forth by Wundt in 1908: "...every objectified content of consciousness, which is a psychological but

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[14] Helmholtz, *Die Thatsachen in der Wahrnehmung: Redegehalten zur Stiftungsfeier der Friedrich-Wilhelms-Universität zu Berlin am 3. August 1878* (Berlin: August Hirschwald, 1879), 6–7; also Warren and Warren, 209. [Das Grundproblem, welches jene Zeit an den Anfang aller Wissenschaft stellte, war das der Erkenntnisstheorie: 'Was ist Wahrheit in unserem Anschauen und Denken? in welchem Sinne entsprechen unsere Vorstellungen der Wirklichkeit?']

[15] Toulmin and Janik's central hypothesis is that Viennese intellectuals, plagued by the uncertainty of their knowledge, "had to face the problem of *the nature and limits of language, expression and communication,*" (p. 117; their emphasis). The Austrian physicist Ernst Mach (1838–1916) took an extreme view, claiming that "the world consists only of our sensations, in which case, we have knowledge only of sensations"; see Ernst Mach, *The Analysis of Sensations and the Relation of the Physical to the Psychical,* trans. C. M. Williams; rev. and suppl. from the 5th German ed. by S. Waterlow (New York: Dover, 1959), 12. Mach's radical empiricism struck a chord among Vienna's artistic elite: Hugo von Hofmannsthal attended Mach's university lectures, Robert Musil wrote a doctoral thesis on Mach, and Carl Schorske mentions Mach and Schoenberg as part of an intellectual tradition that shared "a diffuse sense that all was in flux, that the boundary between ego and world is permeable"; see Carl Schorske, *Fin-de-Siècle Vienna: Politics and Culture* (New York: Vintage, 1981), 345. Mach's interest in music theory was considerable. Besides his lucid *Einleitung in die Helmholtz'sche Musiktheorie: Populär für Musiker dargestellt* (Graz: Leuschner und Lubensky, 1866), he is known to have corresponded with Heinrich Schenker; see Kevin Korsyn, "Schenker's Organicism Reexamined," paper read at the Second International Schenker Symposium, Mannes College of Music, New York, March 1992.

[16] For example, the Austrian language philosopher Fritz Mauthner, in his influential *Beiträge zu einer Kritik der Sprache,* 3 vols. (Stuttgart: J. G. Cotta, 1901–3), used *Darstellung* in connection with sensory representations; see Toulmin and Janik, 133.

fluid event. *Vorstellungen* are formations whose elements are sensations."[17] Riemann we shall see would need to expand this definition.

One senses throughout Helmholtz's writings an implacable resistance to metaphysics. Riemann's assumption of three functional categories was exactly the sort of question-begging that Helmholtz opposed and would not have tolerated in his own work. Helmholtz was particularly wary of Kant's distinction between a priori knowledge ("knowledge absolutely independent of all experience") and empirical knowledge ("knowledge possible only a posteriori, that is, through experience").[18] He believed that all knowledge was a posteriori and that one learned the significance of sense data by drawing "unconscious inferences" ("unbewusster Schlüsse") from their correlations with external events. Kant had argued that this significance was "grounded *a priori* in pure intuition,"[19] and that knowledge was determined through modes of perception that were outside of experience. The idea that perception consisted even partly of innate components—other than the subliminal physiological ones—was objectionable to Helmholtz on several grounds; fundamentally, he believed that the nativism of Kant was "completely without use in explaining our knowledge of the real world, since the propositions produced by it may be applied to the conditions of the real world only after their objective validity has been tested and established empirically."[20]

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[17] Wilhelm Wundt, *Grundriß der Psychologie* (Leipzig: Alfred Kröner, 1908), 404. [jeder objektivierte Bewußtseinsinhalt, der psychologisch aber ein fließender Vorgang ist. Die Vorstellungen sind Gebilde, deren Elemente Empfindungen sind.] Wundt (1832–1920), one of the pioneers of experimental psychology, began his career in Heidelberg as Helmholtz's lab assistant; also see *Eislers Handwörterbuch der Philosophie*, 2d ed., (Berlin: E. S. Mittler und Sohn, 1922), s.v. "Vorstellung."

[18] Immanuel Kant, *Critique of Pure Reason*, trans. N. K. Smith (New York: St. Martin's Press, 1965), 43.

[19] Kant, *Critique of Pure Reason*, 141.

[20] Warren and Warren, 246. Helmholtz introduced the term "nativism" to refer to theories of mind that espoused universal knowledge. Such knowledge was not derived from human

A more congenial relation between Kant and Helmholtz has seemed undeniable to some scholars, perhaps because Helmholtz belonged to a culture that had absorbed so much of Kant's philosophic influence. Helmholtz did pay homage to Fichte on the occasion of his Berlin address in 1878, describing this Kantian (and former rector of the university) as one able to captivate an audience by means of "his bold, soaring concepts of idealistic philosophy,"[21] but these words were surely intended to rouse the sentiments of his own audience. The pervasiveness of idealism in nineteenth-century Germany has dissuaded many from accepting the full extent of Helmholtz's opposition toward Kant. The psychologist Raymond Fancher, for example, has written that the first studies of sensation and perception by Müller and his students were carried out within a "matrix of neurophysiology and Kantian philosophy."[22] Fancher goes on to say that Kant's formulation of noumenal and phenomenal worlds spawned "a climate of opinion in which the scientific study of the mind could develop and prosper."[23] Although Helmholtz sought a physiological explanation where Kant was content to hypothesize abstract mental categories, both distinguished external from internal reality and offered solutions to the problem of perception that Fancher regards as analogous.

If we have gone too far in polarizing Kant and Helmholtz, scholars such as Fancher are overzealous in uniting them, and are wont to play down the genuine conflict between Helmholtz's empiricism and Kant's

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experience, but was innate and intergenerational. Kant used the term "angeboren" in connection with *a priori* forms of knowledge. See *Eislers Handwörterbuch der Philosophie*, s.v. "Angeboren," "Nativismus"; and Dagobert D. Runes, *Dictionary of Philosophy* (New York: Philosophical Library, Inc., 1983), s.v. "nativism."

21 Warren and Warren, 207.

22 Fancher, 91.

23 Fancher, 87.

nativism. One of the motives behind Helmholtz's treatise on physiological optics was to show how the sensations of vision create the idea of space—to affirm, in other words, that space was a learned concept rather than an innate form of intuition.[24] Helmholtz believed that physiological laws played an important role in the formation of higher-level concepts such as space and time, but that experience was the true architect of knowledge.

4.3 Tonal Motion

Melody is the source of music for Helmholtz, and "all melodies are motions within extremes of pitch."[25] Helmholtz discussed tonal motion at length. Of all the arts, he believed that music was uniquely suited to the expression of different kinds and degrees of motion:

> The incorporeal material of tones is much more adapted for following the musician's intention in the most delicate and pliant manner for every species of motion, than any corporeal material however light. Graceful rapidity, grave procession, quiet advance, wild leaping, all these different characters of motion and a thousand others in the most varied combinations and degrees, can be represented [darstellen] by successions of tones.[26]

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[24] Adamant on this point, Helmholtz rejected the conventional wisdom that space was three-dimensional because that was how the mind had to conceive it. Helmholtz showed that after wearing distorting spectacles for a few hours, people began to develop appropriate perceptual responses to non-Euclidean spaces. Kant did not live long enough to witness the emergence of non-Euclidean geometries: Lobatchevsky and Bolyai began to show that such geometries could be used to describe properties of physical space as accurately as Euclidean geometry in 1825. See Morris Kline, *Mathematics: The Loss of Certainty* (New York: Oxford Univ. Press, 1980), 81–88.
[25] Helmholtz, 413; Ellis trans., 250. [Alle Melodie ist eine Bewegung innerhalb wechselnden Tonhöhe.]
[26] Helmholtz, 413; Ellis trans., 250. [Das unkörperliche Material der Töne ist viel geeigneter, in jeder Art der Bewegung auf das Feinste und Fügsamste der Absicht des Musikers zu folgen, als irgend ein anderes noch so leichtes körperliches Material; anmutige

The idea of tonal motion seems to have had both a literal and a figurative sense for Helmholtz. From a physical standpoint, discrete tones corresponded to vibrating bodies or masses, and the differences between them could be measured precisely. Helmholtz liked to compare the imperceptible motions of sound waves to the visible motions of waves of water: Throwing a stone into a pool of water was not unlike setting a sonorous body in motion; the circular waves generated by the stone were analogous to the spherical waves generated by the sonorous body.

More often Helmholtz used the word "motion" metaphorically, to describe the illusion of motion that a series of pitches could conjure in the imagination of a listener. Tonal motion in this figurative sense amounted to "a change of pitch in time,"[27] for notes did not move from one place to another any more than listeners moved from one note to another. The significance of such motions was not only in their association with real motions, but also with the mental conditions that were antecedent to physical actions of "wild leaping," "grave procession," and the like.

Helmholtz was careful to distinguish mental conditions (Gemütszustände), which were expressible in music, from human feelings (Gefühle), which were not. He believed mental conditions to be general states of mind, such as agitation or longing, and characterized them in dynamic terms: "Our thoughts may move fast or slowly, may wander about restfully and aimlessly in anxious excitement, or may keep a

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Schnelligkeit, schwere Langsamkeit, ruhiges Fortschreiten, wildes Springen, alle diese verschiedenen Charaktere der Bewegung und noch eine unzählbare Menge von anderen lassen sich in den mannigfaltigsten Schattierungen und Kombinationen durch eine Folge von Tönen darstellen]

[27] Helmholtz, 416; Ellis trans., 252. [Die melodische Bewegung ist Veränderung der Tonhöhe in der Zeit.]

determinate aim distinctly and energetically in view." Feelings on the other hand, such as love or piety, were emotional states that gave rise to mental conditions. One could speak of the object of a feeling—a feeling *for* someone—but not of a mental condition. According to Helmholtz, experience indicated a fluid, one-to-many relationship between feelings and mental states: Diverse states of mind issued from one and the same feeling, just as diverse feelings stood behind one and the same state of mind. By asserting a freedom of relation between feelings and mental states, and identifying tonal motion with the latter rather than than former, Helmholtz imparted a new clarity to the subject of musical representation:

> Love is a feeling. But music cannot represent it directly as such. The mental states of a lover may, as we know, shew the extremest variety of change. Now music may perhaps express the dreamy longing for transcendent bliss which love may excite. But precisely the same state of mind might arise from religious enthusiasm. Hence when a piece of music expresses this mental state it is not a contradiction for one hearer to find in it the longing of love, and another the longing of enthusiastic piety.[28]

Through tonal motion, Helmholtz believed that one could receive "a more perfect and impressive image of the 'tune' [Stimmung] of another person's mind, than by any other means, except perhaps by a very perfect

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[28] Helmholtz, 415; Ellis trans., 252. [Liebe ist ein Gefühl. Direkt als solche kann sie nicht durch die Musik dargestellt werden. Die Stimmungen eines Liebenden können bekanntlich den äußersten Grad des Wechsels zeigen. Nun kann die Musik etwa das träumerische Sehnen nach überschwänglicher Glückseligkeit ausdrücken, welches durch Liebe hervorgerufen werden kann. Genau dieselbe Stimmung kann aber auch durch religiöse Schwärmerei entstehen. Wenn also ein Musikstück diese Stimmung ausdrückt, liegt kein Widerspruch darin, wenn der eine Hörer darin die Sehnsucht der Liebe, der andere die Sehnsucht frommer Begeisterung findet.]

dramatic representation of the way in which such a person really spoke and acted."[29] For a melody to make any impression, however, listeners had to be able to measure the distance and time between its component pitches. Measurement of time was given by the musical beat—"the regular alternation of accentuated and unaccentuated sounds in music"—whereas the diatonic scale "furnished a means of measuring the intervals..., so that the equality of two intervals lying in different sections of the scale would be recognised by immediate sensation."[30] It is not surprising, given the melodic orientation of his theory, that Helmholtz used the diatonic scale as a model of tonal space. As limiting as this was, Helmholtz was the first to conceive of tonal motion explicitly as motion within a particular tonal space. He went out of his way to draw a parallel between physical and tonal space, the motions within these spaces, and the "motive forces" underlying these motions—in music, the mental states that he claimed were the psychological basis of musical expression:

> Such a close analogy consequently exists in all essential relations between the musical scale and space, that even alteration of pitch has a readily recognised and unmistakable resemblance to motion in space, and is often metaphorically termed the ascending or descending motion or progression of a part. Hence, again, it becomes possible for motion in music to imitate the peculiar characteristecs of motive forces in space, that is, to form an image of the various impulses and forces which lie at the root of motion.

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[29] Helmholtz, 414; Ellis trans., 250–51. [ein vollkommeneres und eindringlicheres Bild von der Stimmung einer anderen Seele gegeben werden, als es durch ein anderes Mittel, ausgenommen etwa durch eine sehr vollkommene dramatische Nachahmung der Handlungweise und Sprechweise des geschilderten Individuums, geschieht.]

[30] Helmholtz, 596; Ellis trans., 370. [ein Maß für die Abstände der Töne in derselben gegeben wird, so daß wir in unmittelbarer Empfindung zwei gleiche Intervalle, die in verschiedenen Abschnitten der Leiter liegen, als gleich anerkennen.]

> And on this, as I believe, essentially depends the power of music to picture emotion.[31]

Tonal distance depended not only on the diatonic relationship between pitches, but on their acoustic relationship as well. Helmholtz made a broad distinction between first-degree and second-degree acoustic relations: "We shall consider musical tones to be related in the first degree which have two identical partial tones; and related in the second degree, when they are both related in the first degree to some third musical tone."[32] There are no examples of second-degree acoustic relations in Helmholtz's writings, but let us take his definition at face value for the moment. The pitches C3 and G3 are related in the first degree because the third and sixth partials of C3 (G4 and G5, respectively) are also the second and fourth partials of G3. Similarly, the third and sixth partials of F3 (C4 and C5, respectively) are the second and fourth partials of C3. Since F3 and G3 each relate to C3 in the first degree, they would by definition relate to each other in the second degree.[33]

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31 Helmholtz, 597; Ellis trans., 370. [Dadurch ist in wesentlichen Verhältnissen eine so große Ähnlichkeit der Tonleiter mit dem Raum gegeben, daß nun auch die Änderung der Tonhöhe, die wir ja oft bildlich als Bewegung der Stimme nach der Höhe oder Tiefe bezeichnen, eine leicht erkennbare und hervortretende Ähnlichkeit mit der Bewegung im Raum erhält. Dadurch wird es weiter möglich, daß die musikalische Bewegung auch die für die treibenden Kräfte charakteristischen Eigentümlichkeiten der Bewegung im Raum nachahmt und somit auch ein Bild der der Bewegung zugrunde liegenden Antriebe und Kräfte gibt. Darauf wesentlich beruht, wie mir scheint, ihre Fähigkeit, Gemütsstimmungenauszudrücken.]

32 Helmholtz, 423–25; Ellis trans., 256–57. [Verwandt im ersten Grade nennen wir Klänge, welche zwei gleiche Partialtöne haben; verwandt im zweiten Grade solche, welche mit demselben dritten Klang im ersten Grade verwandt sind.] When Helmholtz speaks of identical partial tones, he means absolute pitches rather than pitch classes. Helmholtz does not give an example of second-degree relations, and it is difficult to be sure of his meaning.

33 Given a pure interval with frequency ratio $p : q$, the npth partial of the lower tone will coincide with the nqth partial of the higher tone (where n is a positive integer). Thus, in the case of the fifth C3–G3 (2 : 3), the third partial of C3 coincides with the second partial of G3, the sixth partial (2 · 3) of C3 coincides with the fourth partial of G3 (2 · 2), and so on.

Of course, any pure interval will satisfy the definition of a first-degree relation if one looks far enough along the harmonic series of the two pitches. Consider the "second-degree" relation F3–G3 (8 : 9); the ninth and eighteenth partials of F3 coincide with the eighth and sixteenth partials of G3, which means that the pitches are related in the first degree if one is willing to accept—in the abstract, for they cannot be heard—these higher partials. Helmholtz gives no indication as to how far along the harmonic series he will go to assert a first-degree relation between two pitches; just as consonance and dissonance are relative concepts for him, first-degree and second-degree relations are more blurred than their definition might suggest. Helmholtz does say that tones related in the first degree are especially close if their common partials occur early in the series. Closeness of relationship, in other words, depends on the loudness of the coincident partial tones. Since coincident partial tones having a higher ordinal number in the overtone series are generally weaker than those with a lower ordinal number, the relationship of two tones becomes more remote as the ordinal numbers of the coincident partials increase. Using this principle, Helmholtz ranks the notes in the octave above C3 according to the closeness of their first-degree relationship with C3. He arrives at the following series of notes: C4–G3–F3–A3–E3–E^{b}3. He does the same for the octave below C3, and arrives at an inverse series: C2–F2–G2–E^{b}2–A^{b}2–A2.[34]

See Rudolph Rasch, "Theory of Helmholtz-Beat Frequencies," *Music Perception* 1/3 (1984): 308–22.

[34] Helmholtz, 425; Ellis trans., 257. These rankings raise questions about the viability of Helmholtz's distinction between first-degree and second-degree relations. For each of the pitches A3 and E3, one must look beyond the senario to fulfill the first-degree criterion: The sixth and seventh partials of A3 coincide with the fifth and sixth partials of C3, and the fourth and eighth partials of E3 coincide with the fifth and tenth partials of C3. Even if we accept the first-degree relation between A3 and C3, and E3 and C3, it does not follow

Chordal relation is analogous to tonal relation. Depending on the physical correspondence between two chords, they are either directly related, or related in the second degree. Helmholtz restricts his discussion to major and minor triads, which for him are the only consonant chords. Chords that have one or more tones in common are directly related. Chords that are directly related to a third chord are related to each other in the second degree. Direct relations would include C major and G major, or C major and A minor.[35] G major and A minor, however, are second-degree relations since both are directly related to C major and neither shares common tones with the other. Helmholtz distinguishes direct chordal relations with one common tone from those with two common tones. Chords with two tones in common are more closely related than chords that share just one tone. C major and A minor, therefore, are more closely related than C major and G major. Matthew Shirlaw disputes this statement from the standpoint of common practice tonality.[36] Helmholtz, however, is claiming a physical relation between sounds, not a hierarchical relation between chord functions.[37]

A conflict that Helmholtz does leave unresolved in his discussion of chords is the incompatibility of direct chordal relations with first-degree tonal relations. When speaking of chords, fifth relations are always more remote than third relations for Helmholtz, if third-related chords share two common tones. Third-related tones, however, are always more

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in this case that the pitches "related in the first degree to some third musical tone" relate in the second-degree relation to each other.

[35] Helmholtz, 481; Ellis trans., 296. Tuning is relevant here: Helmholtz writes that c–e–g and g–b–d are directly related, as are c–e–g and a–c–e.

[36] Matthew Shirlaw, *The Theory of Harmony.* (London: Novello, 1917; reprint, Dekalb, Ill.: B. Coar, 1955), 106–7.

[37] Present-day psychoacousticians claim that the dominant triad is more closely related than the submediant to the tonic.

remote than fifth-related tones because their coincident partials are higher in the overtone series. Helmholtz's use of one criterion to formulate tonal relations and another to formulate chordal relations results in friction between the tonal and chordal levels of his pitch space.

Helmholtz scarcely mentions key relations. The abstract nature of keys may best explain their scant treatment, since keys, unlike chords and tones, do not lend themselves to "side-by-side" physical comparison. Helmholtz casually states that one may modulate with good effect to closely related keys a fifth or third away from the tonic. These recommendations merely extend the tonal and chordal relations he outlines to the level of key. Ellis appends a long essay on temperament to his translation of Helmholtz, in the course of which he formulates a table of modulation whose rows are major-third cycles and columns are perfect-fifth cycles. The table is adapted to just intonation and is probably a reasonable approximation of Helmholtz's own views of key relation.[38]

[38] Ellis trans., 430–556. Ellis's table, or Duodenarium, appears on p. 463. The text beneath the Duodenarium reads: "This is the first table of modulations adapted to Just Intonation that has been constructed. But the table in Gottfried Weber's *Versuch einer geordneten Theorie der Tonsetzkunst...*, although only adapted to equal temperament, was of much assistance to me." It is worth noting that the Duodenarium appeared nine years after the Table in Oettingen's *Harmoniesystem*, and that Weber's pitch space is substantially different from Ellis's: The columns of Weber's space are perfect-fifth cycles, but the rows pair major and relative minor keys in cycles of minor thirds: E^{b} c, C a, A $f^{\#}$, etc.; see Weber, 3d rev. ed., vol. 2, 86.

4.4 Tonal Representation

The idea of tonal representation preoccupied Riemann in his last two theoretical essays, "Ideen zu einer 'Lehre von den Tonvorstellungen'" and "Neue Beiträge zu einer 'Lehre von den Tonvorstellungen.'" In these essays Riemann focused entirely on music perception. Our comments refer to the first essay, the more important of the two, which we shall simply call "Ideen."

In "Ideen" Riemann expressed his disappointment with tone psychology, particularly as evidenced in the work of Helmholtz and Stumpf.[39] Both Helmholtz and Stumpf felt that to understand music perception one first had to understand the perception of single tones. Riemann believed this bottom-up enterprise was misguided and had forestalled broader, synthetic approaches to music perception. As a corrective, he reasserted the notion of *Tonvorstellung* upon which he hoped to ground music psychology, and spoke of the "important role that the representation of tones and tonal motions have long had for the musician."[40] What did Riemann mean by tonal representation and motion? One psychologist has suggested that Riemann was saying no more than Helmholtz, namely that tonal motions were manifestations of

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[39] Carl Stumpf (1848–1936) was an experimental psychologist known for his study of intervallic fusion, or *Tonverschmelzung*. Stumpf isolated various intervals and asked subjects whether they perceived these as sonic wholes or as composites. Riemann hoped that Stumpf's research would hasten the development of music psychology, and lamented that the "logische Aktivität" of hearing, which he had advocated as early as 1873, played no role in Stumpf's two-volume *Tonpsychologie* (Leipzig: S. Hirzel, 1883–1890); see Katherine Arens, *Structures of Knowing: Psychologies of the Nineteenth Century*, Boston Studies in the Philosophy of Science, vol. 113 (Dordrecht: Kluwer Academic Publishers, 1989), 26–27.

[40] Riemann, "Ideen," 3. [...welch wichtige Rolle das Vorstellen von Tönen und Tonbewegungen schon längst, nein immer, und zu allen Zeiten für das Musizieren gespielt hat und spielen muß.]

the interior motions of our minds.[41] Riemann indeed said this in a reference to Herder's aesthetics in 1900; however, he was concurring with that philosopher's poetic notion that "music sounds a clavichord within us all, our innermost nature," rather than making a psychoacoustic claim about music perception.[42] For both Riemann and Helmholtz tonal sensations—Wundt's "objectified contents of consciousness"—were the basis of tonal representations; but for Riemann the *Vorstellung* concept only began here. Essential to his conception were the interpretive faculties that animated or "subjectified" sense data. Riemann stated in the opening sentence of "Ideen" that "listening to music is not merely a passive sensation of sound waves in the ear, but a highly developed operation of logical functions of the human mind."[43] His *Vorstellungen* were, in short, products of an active imagination working on passively received sense data.

Helmholtz conceptualized tonal motion in melodic terms. Neither chords nor key areas, but, rather, pitch successions mimicked the physical activities of the mind. For Riemann, tonal motion was a harmonic concept. This was due to his belief that pitches were meaningful only in connection with specific chords. Riemann had developed this idea earlier, in "Die Natur der Harmonik," under the rubric of *Tonvertretung* by which he meant the proxy "representation" of triads by individual pitches.

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41 Robert Francès, *The Perception of Music*, trans. W. Jay Dowling (Hillsdale, N. J.: Lawrence Erlbaum Associates, Publishers, 1988), 307.

42 Riemann, *Musikalische Aesthetik* (Berlin und Stuttgart: W. Spemann, 1900), 21. Riemann writes: "Herder perceived clearly that tonal motions are an image of the motions of our mind." [Herder erkennt deutlich, daß die Tonbewegungen ein Abbild der Bewegungen unserer Seele sind.] Herder's own "Die Musik spielt in uns ein Klavichord, das unsere eigene, innigste Natur ist" is cited on the same page. See also Francès, 307 n. 21.

43 Riemann, "Ideen," 1. [Daß das Musikhören nicht nur ein passives Erlieden von Schallwirkungen im Hörorgan sondern vielmehr eine hochgradig entwickelte Betätigung von logischen Funktionen des menschlichen Geistes ist,...]

Depending on the musical context, an unaccompanied pitch could stand for one of three major or minor triads: C could stand for F–A–C, A^{b}–C–E^{b}, or C–E–G, or for F–A^{b}–C, A–C–E, or C–E^{b}–G. Riemann extended the *Vertretung* concept to isolated triads, which themselves could be thought of in any one of three major or minor keys.[44] Tonal motion was thus a species of triadic motion, and modulation was—psychologically speaking—a motion from a state of relative consonance to one of dissonance.

In "Ideen" Riemann took his psychological approach further and dealt with tonal motions and representations that were independent of actual musical performance. This emancipated view of *Vorstellungen* grew, he claimed, out of years of study of the music of Beethoven. Riemann now believed that "the alpha and omega of music are not to be found in actually sounding music, but in the newly arising representations of tonal relations in the imagination of the creating artist, before they are notated, and again in the imagination of the listener."[45] William Mickelsen has described Riemann's theory of *Tonvorstellung* as including: 1) the ability of the mind to hear or imagine music not actually being performed; 2) the ability of the mind to interpret the logic or illogic of musical activities on the basis of past experiences.[46] Helmholtz's *Vorstellungen* were neither independent of experience, nor products of logical analysis.

The mind created tonal representations according to what Riemann called the "law of greatest possible economy." The greatest economy of

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[44] Riemann, "The Nature of Harmony," 28.
[45] Riemann, "Ideen," 2. [... daß nämlich gar nicht die wirklich erklingende Musik sondern vielmehr die in der Tonphantasie des schaffenden Künstlers vor der Aufzeichnung in Noten lebende und wieder in der Tonphantasie des Hörers neu erstehende Vorstellung der Tonverhältnisse das Alpha und das Omega der Tonkunst ist.] (Riemann's emphasis).
[46] Mickelsen, 86.

representation was one that rejected complicated *Tonvorstellungen.* For instance, the mind would construe a pitch as part of a major or minor triad before attempting more roundabout interpretations. Riemann did not say whether root interpretations were more likely than third or fifth interpretations, but left it to musical context to determine the proper judgment. As a further example of psychological economy, Riemann claimed that chords derived from the senario were consonant because the senario contained the simplest numbers. From this it followed that one would prefer root progressions by perfect fifth and major third because 2 : 3 and 4 : 5 were the simplest ratios after the octave (3 : 4 merely inverts 2 : 3). Tonal organization both in creating and listening to music resulted from "the quest for simplicity."[47]

Consolidation was a vital impulse in theories of all kinds at the turn of the century, and Riemann's principle of economy may be viewed as part of a larger trend toward synthesis, and away from the analytic methods of scientists such as Helmholtz, Mach, and Stumpf. In the first decades of the twentieth century, around the time Riemann was writing his last essays, psychologists began to consider new approaches to the study of perception. Three young German psychologists—Max Wertheimer, Kurt Koffka, and Wolfgang Köhler—took great interest in the degree to which the mind simplified experience by organizing it into indivisible wholes or *Gestalts.* The Gestalt psychologists, like Riemann, formulated a handful of working hypotheses about perception. These hypotheses did not presume to explain the mechanics of perception, but instead provided a conceptual model or *Darstellung* by means of which individual perceptions might be

[47] Mickelsen, 86. Riemann does not linger over his criteria for simplicity. There is no discussion, for example, of why only the numbers one through six are simple.

better understood. Riemann's hypotheses were similar to those advanced by the Gestalt psychologists: The "law of greatest possible economy" prefigured the Gestalt principle of *Prägnanz,* the theory of harmonic functions invoked psychological principles of categorization, and the Table of Relations provided a measure of "good continuation" in music.

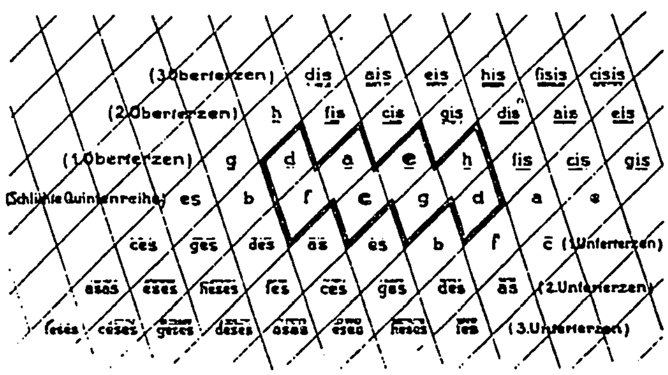
Whereas the tone psychologists had worked inductively, drawing general conclusions from a mass of specific observations, Riemann worked in the opposite direction, drawing conclusions about individual perceptions from general hypotheses. His contemporary, the Austrian physicist Ludwig Boltzmann (1844–1906), had much to say about the hypothetical character of scientific theories, and described a process of "deductive representation" that might have struck a resonant chord:

> ... at first we operate with thought abstractions, mindful of our task only to construct inner representation-pictures. Proceeding in this way, we do not as yet take possible experiential facts into consideration, but merely make the effort to develop our thought-pictures with as much clarity as possible and to draw from them all possible consequences. Only subsequently, after the entire exposition of the picture has been completed, do we check its agreement with experiential facts.... We shall call this method deductive representation.[48]

[48] Ludwig Boltzmann, "Theories as Representations," in *Philosophy of Science,* ed. A. Danto and S. Morgenbesser (New York: Meridian Books, 1960), 249; also Ludwig Boltzmann, *Populäre Schriften,* 3d ed. (Leipzig: J. A. Barth, 1925), 261–62. [...daß wir eingedenk unserer Aufgabe, bloß innere Vorstellungsbilder zu konstruieren, anfangs lediglich mit gedanklichen Abstractionen operieren. Hierbei nehmen wir noch gar keine Rücksicht auf etwaige Erfahrungstatsachen. Wir bemühen uns lediglich, mit möglichster Klarheit unsere Gedankenbilder zu entwickeln und aus denselben alle möglichen Konsequenzen zu ziehen. Erst hinterher, nachdem die ganze Exposition des Bildes vollendet ist, prüfen wir dessen übereinstimmung mit den Erfahrungstatsachen.... Wir wollen dies als die deduktive Darstellung bezeichnen.]

In the fourth part of "Ideen," which included a discussion of tone, chord, and key relation, Riemann presented the Table of Relations. We have reproduced this *Darstellung* of Riemann's tonal universe in Example 4-1. A brief account of Riemann's use of the Table to represent tonal motion and distance in accordance with the law of greatest economy follows.

EXAMPLE 4-1: THE TABLE OF RELATIONS IN "IDEEN"



Riemann begins his discussion of the Table in "Ideen" by observing that subdominant, dominant, relative, and parallel relations determine the closest distances. These relations are all clear, with the exception of the relation between parallel major and minor. Riemann says that one might easily be misled into looking for C minor in the vicinity of E^{b} major, its relative major, whose tonic lies -3Q from the central C. One needn't go this far afield to find the parallel key, however, for C minor is simply the flip side of C major. Since the diagonals allow Riemann to represent all major triads in a given horizontal series with upright triangles, all minor triads may be represented with inverted triangles, and the relation between tonics of parallel major and minor keys may be represented as a parallelogram. Besides perfect fifths and major thirds, the verticals describe chromatic half-step relations, and the southeast diagonals describe minor thirds. The framed middle section contains the pitches of the diatonic scale, C major or A minor, with both values of D (the subdominant 'd' in A minor, and the dominant 'd' in C major).

Although Riemann mentions the Table's minor third and chromatic half-step series, he exploits only the perfect fifths and major thirds. Through these intervals, he says, the Table divulges the simplest path between two pitches. Most of the remainder of the fourth section of "Ideen" is given over to a formalization of these relations, which we shall take up at length in the next chapter. Riemann introduces the symbols Q and T to denote upper fifth (*Quint*) and third (*Terz*) relations. Their inverses $^1/_Q$ (-Q) and $^1/_T$ (-T) denote lower fifth and third relations. With these symbols, Riemann systematically formulates the distances among keys on the Table.[49] For example, A major is $^T/_Q$ rather than 3Q from 'c', because the major triad on 'a' (3Q from 'c') possesses no tones in common with the framed pitches in Example 4-1; the major chord on 'a' ($^T/_Q$, or -Q + T) contains two common tones ('a' and 'e') and is therefore judged to be closer to the reference tonality. For the same reason, C minor (whose tonic is 'g') is $^Q/_T$ rather than $^1/_{3Q}$ from 'e' (the tonic of A minor). In obeisance to the "law of greatest possible economy," Riemann always prefers the simpler formalization.

[49] Riemann's use of Q and T bears comparison with David Lewin's more recent use of D and M (dominant and mediant) to formalize tonal functions. Lewin goes one formal step further than Riemann by considering values for his D and M other than perfect fifths and major thirds; see David Lewin, "A Formal Theory of Generalized Tonal Functions," *Journal of Music Theory* 26/1 (1982): 23–60.

4.5 Summary

Tone psychology and music psychology were both committed to explaining the perception of aural phenomena. In this respect, the difference between Helmholtz and Riemann was one of method rather than aim. Elizabeth West Marvin generalizes this sentiment when she states that tone psychologists and music psychologists differed mainly in the musical stimuli they chose to study.[50] Tone psychologists believed that by divesting musical elements of their contexts they would be better able to formulate general laws of perception. Simplifying the stimulus, in other words, would yield the laws by which one would later come to understand dynamic musical contexts. Helmholtz laid the foundations of this approach. In the aftermath of Stumpf's work, however, it became clear that tone psychology was moving too slowly and that the inductive method was not achieving results. Riemann believed that rather than simplify the object of investigation, it would be more profitable to start with general hypotheses that could be tested on actual pieces of music.

We have suggested that the approaches taken by Helmholtz and Riemann were symptomatic of broader theoretical trends in nineteenth-century philosophy and science. For Helmholtz, judgments of tonal distance were tied to empirical sense data; for Riemann, they rested on hypotheses such as the Table of Relations. We have claimed that the rise of Gestalt psychology in the early years of this century coincided with a similar elevation of general hypotheses over specific sense data.

Although both theorists spoke of *Vorstellungen*, Riemann's sense of the term was more inclusive than Helmholtz's, encompassing sense data

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[50] Marvin, 83.

as well as their interpretation through higher-level cognitive strategies. Marvin says that even though Riemann believed listeners employed hypothetical "sound-structures" (*Klanggebilde*) when listening to pieces of music, he was never able to formulate a representational model for these structures.[51] The Table of Relations was just such a model. Before music psychology could hope to get off the ground, music had to be thought of as more than a series of discrete acoustic events. Riemann took this step by conceiving a system of musical functions, with a logic or grammar of their own, and by addressing music perception at an encompassing level where this grammar or logic played itself out.

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[51] Marvin, 72.

CHAPTER 5: THE SYSTEM OF HARMONIC SCHRITTE AND WECHSEL

5.1 Introduction

Riemann's shift from musical logic and the *große Cadenz* to musical syntax and the Table coincided with a broader shift from speculative to practical music theory. This practical impulse found its first expression in *Skizze einer neuen Methode der Harmonielehre* (1880), and remained a vital force in Riemann's work for nearly forty years. *Skizze* was less austere and more successful than anything Riemann had published to date. The chord connective approach of *Musikalische Syntaxis* was here expanded into a thorough, well-organized, and accessible *Harmonielehre.* Mindful of his reader (not to mention his publisher), Riemann traded in the tongue-twisting language of earlier work for a friendlier (German) and more concise terminology: Antilogic and homologic relations were merged into a single category of harmonic *Schritte* (steps), antinomic relations were redefined as harmonic *Wechsel* (changes of mode), and homonomic relations were left undefined as the default case. Whereas *Musikalische Syntaxis* had offered no relief from its dense text, and had burdened the reader with a surfeit of *Klangschlüssel, Skizze* offered a simpler language and a liberal use of examples. Here at last was a book that musicians might read.

In this chapter we shall pick up the thread of our discussion of *Musicalische Syntaxis* (see 3.13–3.14) and trace the development of Riemann's "Systematik der Harmonieschritte." This ought to have been called the system of *Schritte* and *Wechsel,* since both transformation types were basic to Riemann's conception of harmony. There are several intertwined aspects of the "SW-system" that we shall address in this chapter.[1] First, there is Riemann's symbolic notation, which crystalizes the main features of his harmonic conception. Second, there are the *Schritte* and *Wechsel* themselves, which we treat in connection with the Table. Third, there is the relation between the SW-system and the theory of harmonic functions. Finally, we consider some formal characteristics of the SW-system, which cast the Table in a new light and raise questions about enharmonic equivalence and symmetry in music. In this last section, we shall abstract what mathematicians call group structure from the SW-system and—with the help of the Table—explore this structure along several "transformational axes." This last aspect (which will become plainer as we proceed) is somewhat removed from Riemann's thinking, but draws closely upon his theory.

Thus, we deal first in particulars—the *Schritte* and *Wechsel*, along with their associated symbols—and then in abstraction. Our chief sources for the particular aspects of Riemann's system are the second and sixth editions of *Skizze* (1887; 1918). The generalization that follows is inspired by recent work by Lewin (1987), Hyer (1989), and Klumpenhouwer (1994).

5.2 The *Skizze/Handbuch* Series: An Overview

Skizze went through six editions in Riemann's lifetime. From the second edition onward the work was published as *Handbuch der Harmonielehre*, but preserved in most respects the organization and goals

[1] We use "SW-system" throughout as an abbreviation for the system of *Schritte* and *Wechsel*, or what Riemann called the "Systematik der Harmonieschritte."

of the original.[2] If the change in name signaled a more finished state, this was true mainly in terms of the comprehensiveness of later editions; with each new edition came new *Schritte* and *Wechsel*, until the repertoire of harmonic moves was such that any two major or minor chords could follow each other. Such additions fed a theory that, on its surface, began to affirm Oettingen's "chaos of possibilities." The difference, however, was that Riemann's expansion of syntax was a gradual and deliberate attempt to accommodate nineteenth-century practice. Such moves as the *Gegenterzwechsel* (c+ – °a^{b}) or the *Tritonusschritt* (c+ – f$^{\#}$) should be judged in the light of this practice, and not solely as products of Riemann's imagination. On the other hand, the relation between theory and practice is undeniably oblique in Riemann's work. It is striking and somewhat depressing that a musical polyglot of Riemann's stature should have so little to say about the music of his own time. Riemann generally prefered made-up *Notenbeispielen* in his harmony texts to examples taken from the literature; *Skizze/Handbuch* contained a profusion of these textbook examples but not one single reference to music.[3]

Still, the *Skizze/Handbuch* series is a lucid chronicle of Riemann's mature harmonic theory. Besides charting the field of *Schritte* and *Wechsel*, these works give, from the third edition (1897) onward, up-to-date coverage of the function symbols introduced in *Vereinfachte Harmonielehre* (1893). In the remainder of this chapter we shall use *Skizze* in reference to the second edition—this title did appear in small

[2] All six editions were published in Leipzig by Breitkopf und Härtel: 1/ 1880; 2/1887; 3/1898; 4/1906; 5/1912; 6/1918. By 1929, ten years after Riemann's death, the work was in its 10th edition.
[3] Riemann's *Systematische Modulationslehre als Grundlage der Musikalischen Formenlehre* (Hamburg: J. F. Richter, 1887) is exceptional in this regard; it includes numerous citations from the literature and some musical analysis. See n. 45 below.

print on the work's title page[4]—and *Handbuch* in reference to the sixth. The second edition was a "sketch" insofar as it preceded the watershed material of *Vereinfachte Harmonielehre,* whereas the sixth edition possessed the detail and greater authority of a "Handbuch." We shall not discuss *Vereinfachte Harmonielehre,* since it is well known to most American theorists—it remains the only one of Riemann's major works available in English translation—and because it is incomplete from the standpoint of Riemann's later work. *Skizze/Handbuch* provides the historical and theoretical scope we desire; its six editions, published over four decades, unfold a broad two-fold program: the adaptation of "musical syntax" (this itself an adaptation of Oettingen's *Harmoniesytem*) to the SW-system, and the merging of this latter system with the theory of harmonic functions. In preparation for our discussion of the SW-system, we must grapple first with Riemann's symbolic notation.

5.3 Riemann's Notation and Figured Bass

Riemann's notation is both lock and key to his mature harmonic theory. It is difficult to learn, but fluency is what one needs to appreciate the underlying harmonic conception. Riemann actually developed two kinds of notation—*Klangschlüssel* and *Funktionbezeichnungen*—both of which we have mentioned but have yet to examine in detail. Among the difficulties of Riemann's notation is the fact that both types were under

[4] The title page reads, "Handbuch der Harmonielehre von Dr. Hugo Riemann, Zweite, vermehrte Auflage der 'Skizze einer neuen Methode der Harmonielehre'. Mit Notenbeispielen." Both works bore the dedication "Den Manen Franz Liszt's" (Liszt had died in 1886), which may reflect Riemann's new practical outlook. *Musikalische Syntaxis,* by way of contrast, had been dedicated to "Herrn Dr. Arthur von Oettingen, Professor der Fysik an der Universität Dorpat."

constant revision. Furthermore Riemann did not consistently separate the two types, leaving behind a "hybrid" with features of both. Before crossing into this world of locks and keys, it is imperative to ask why Riemann spent so many years working out this aspect of his theory. The concise answer is that he wanted his notation to replace figured-bass notation; this point deserves expansion, however, since it casts light onto the shadowy relation between Riemann's theory and musical practice.

Figured-bass theory remained a pillar of music education throughout the nineteenth century. Riemann felt that the holdover of figured bass, while justifiable for the study of earlier music, was at odds with contemporary practice. Figured-bass theory perpetuated a linear view of harmony at a time when harmonic practice was uniquely—even radically—chordal. Taking his lead from Weber (and Vogler before him), Riemann set out to reform harmonic theory. The issue was not whether figured bass was adaptable to nineteenth-century harmonic practice (clearly it was), but whether it was appropriate—aesthetically and pedagogically—to view nineteenth-century music through the theoretical lens of an earlier era. Riemann felt it was not. Change was needed, not for its own sake (*Neuerungssucht*) but in order to preserve a harmonious relation between theory and practice ("die Theorie und Praxis in Einklang zu erhalten").[5]

[5] Riemann stresses the relation between theory and practice in the *Vorwort* to the first edition of *Skizze*: "I can maintain in good conscience that my proposals have arisen not for their own sake, but from a continuing endeavor to reconcile theory and practice, and to draw upon everything that can enlighten the student in the quickest manner as to the harmonic structure of musical works. May one regard them so, and not discard them out of hand as heretical." [Mit gutem Gewissen darf ich behaupten, dass meine Vorschläge nicht der Neuerungssucht entsprungen sind; dieselben sind vielmehr das Resultat andauernden Bestrebens, die Theorie und Praxis in Einklang zu erhalten und alles heranzuziehen, was den Harmonieschüler am schnellsten über die harmonische Struktur musikalischer Werke aufklären kann. Möge man sie so auffassen und nicht als ketzerisch ungeprüft verwerfen.] This qualification of motive—"Einklang" rather than "Neuerungsucht"—has the ring of

At the same time, Riemann saw the versatility of figured-bass notation and realized he would need an equally durable notation if his ideas were to hold sway. This notation would embody what he termed the "Lehre von den Klängen," a doctrine based upon the composite nature of sound (*Klang*), and stressing chord (*Akkord*) over simultaneity (*Zusammenklang*) and melodic line.[6] The concepts of *Klangvertretung* and harmonic dualism were basic to this outlook, and obvious catalysts in the push toward a new notation. *Klangvertretung*—the part-for-whole relation between pitches and chords—forced questions concerning the very purpose of notation. Unlike figured bass, which presumed a correspondence between figures and actual notes, *Klangvertretung* presumed a notation capable of "figuring" virtual or implied notes. This notation would assert hypothetical *Ober-* or *Unterklänge* for pitches and

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appeasement. Breitkopf und Härtel probably dreaded another excursion along the lines of *Musikalische Syntaxis*, and may have viewed Riemann as a risky prospect. They did not publish his "Musikalische Grammatik," and were still awaiting "Neue Schule der Harmonik" when *Skizze* went to press (see Chap. 3 n. 3). It was in Riemann's interest to present his ideas as conservatively and accessibly as possible.

6 *Klang*, *Akkord* and *Zusammenklang* are distinct concepts whose outlines are often dulled in translation. *Klang* is the general term; it denotes sound in its natural state, which for Riemann means a fundamental pitch plus its overtone and undertone series. *Akkord* is used in connection with major and minor triads, the two chord types that Riemann recognizes as consonant. It is illogical to speak of major or minor *Klänge*, since both chord types are contained in any given *Klang*. *Akkord* is the proper limiting term, but Riemann will sometimes refer to *Klänge* "in the major sense" (*Oberklang*) or "in the minor sense" (*Unterklang*). *Klänge*, viewed as sets, possess infinite cardinality, since overtones and undertones are parts of nonterminating series. In practice, however, Riemann restricts the concept of *Klang* to the first five partials above and below a fundamental pitch. Under this restriction, the *Klang* with C4 as fundamental has a cardinality of 11 and contains the pitches {F1, A♭1, C2, F2, C3, C4, C5, G5, C6, E6, G6}. If we assume octave equivalence (as Riemann does), we may reduce the concept further and write {F3, A♭3, C4, E4, G4}. In the remainder of this chapter, we shall understand the term *Klang* in this reduced sense. It is easy to see that any *Klang* contains just two *Akkorde*—one major and one minor—whose intersection is the fundamental and whose union is the *Klang* itself. The third term, *Zusammenklang*, will be of less concern in what follows. This "sounding together" describes the by-product of concurrent melodic lines; "Simultaneity" or "linear harmony" are possible translations. The Tristan chord would be an instance of *Zusammenklang* for Riemann; the vertical sonority is inexplicable as a *Klang*, and must be viewed as an "accident" of the rhythmic displacement of several melodic strands.

intervals whose chordal meanings were unclear. We shall see how *Klangvertretung* as well as dualism and harmonic function shaped Riemann's symbolic notation. Our preliminary remarks have touched upon another of Riemann's broader ambitions, however—the desire for an interpretive rather than descriptive harmonic theory. Figured-bass notation was not just archaic, it was without means to qualify the relationships between chords. A string of figures revealed no more about chord relationship than a string of unfigured chords. Through his notation, then, Riemann had the dual aim of delivering music theory into the nineteenth century and fostering an interpretive stance toward music in general. He expressed the timeliness of his reform in the foreword to the first edition of *Skizze*:

> In no way do I conceal from myself the difficulties posed by the revision of the pedagogy of harmony. It seems certain, though, that such revision is a question for our time. In times earlier, when a chord was viewed merely as the chance encounter of several voices, figured-bass was the only possible method of notation. Today, when the "Lehre von den Klängen" and *Klangvertretung* have become common property, we fairly demand a notation that allows the significance of chords to be understood.[7]

[7] Riemann, *Skizze* (*Vorwort* to first edition), n.p. [Keineswegs verhehle ich mir die Schwierigkeiten, welche sich jeder Neugestaltung der Harmonie-Lehrmethode entgegenstellen; dass aber eine solche Neugestaltung eine Frage unserer Zeit ist, scheint mir zweifellos festzustehen. Für jene frühere Zeit, welche im Akkord nur ein zufälliges Zusammentreffen mehrerer Stimmen sah, war die Generalbassschrift die einzig mögliche Art der Akkordbezifferung; heute, wo die Lehre von den Klängen und der Klangvertretung Gemeingut geworden ist, dürfen wir auch nach einer Bezifferung verlangen, welche die Klangbedeutung der Akkorde erkennen lässt.]

5.4 *Klangschlüssel*

Riemann's initial response to figured-bass notation was a system of chord labels he called *Klangschlüssel.* This was in some ways a dualist counterpart to figured bass and, as such, shed scarcely more light on the "significance of chords" than its predecessor. The *Schlüssel,* or "ciphers," denoted the content of chords but not their function. On the other hand, Riemann made distinctions with this notation that were tentative or completely undeveloped in figured-bass notation. Most importantly, the *Klangschlüssel* reflected and reinforced his triadic view of harmony. According to this view, major and minor triads—the structural complements of the *Klang*—were the building blocks of musical comprehension. Individual pitches or intervals were meaningful only as extrapolations of major or minor chord templates.[8] Harmonies consisting of four or more pitches were likewise extrapolations, but involving more than one template. The dominant seventh of C major, for example, was formed from the triads G–B–D and F–A–C; in this case G–B–D was complete in the chord, whereas F–A–C was an extrapolation of the pitch F. As the building blocks of this system, major and minor triads were also its sole consonances. Since pitches and intervals had no independent meaning, they were consonant only insofar as they stood (through the principle of *Klangvertretung*) for individual major or minor triads. Chords with four or more distinct pitches were dissonant by default. We shall have more to say about consonance and dissonance below; in

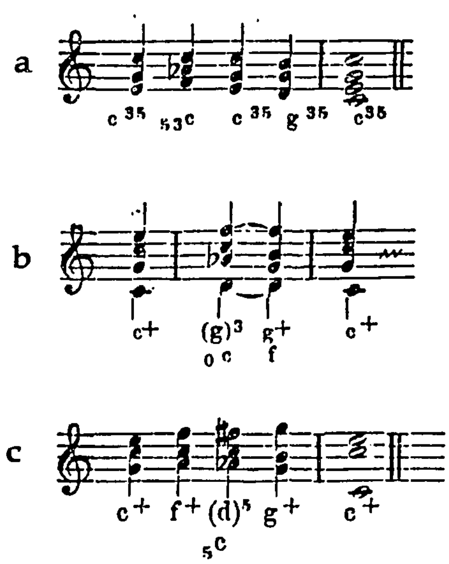
[8] Harmonic templates and their relation to musical consonance are treated in Ernst Terhardt, "The Concept of Musical Consonance: A Link Between Music and Psychoacoustics," *Music Perception* 1/3 (1984): 276–95. Riemann does not use the word "template," but his conception is similar to that of Terhardt.

notational terms, the *Klangschlüssel* supported Riemann's notion of musical consonance by identifying the triadic rather than intervallic components of chords. In its basics, this notation retained the features of that used by Oettingen in *Harmoniesystem*: Lowercase letters stood for chordal roots; '+' and '°' stood for major and minor, respectively. 'g+' was thus equivalent to a G major chord (in any inversion), and '°d' was equivalent to a G minor chord (in any inversion). The important differences between *Klangschlüssel* and figured-bass notation may be summarized as: 1) *Klangschlüssel* isolate chordal roots, figured-bass symbols do not; 2) *Klangschlüssel* indicate triads, figured-bass symbols indicate intervals; 3) *Klangschlüssel* indicate chord quality, but not inversion. Figured-bass symbols with roman numerals indicate inversion, but not chord quality; 4) *Klangschlüssel* distinguish consonance (the underlying triad 'g+', '°d',...) from dissonance (figures added to this triad). Figured-bass symbols do not distinguish consonant from dissonant chord tones.

The above differences are due to a more fundamental difference between *Akkorde*, the subject of *Klangschlüssel*, and *Zusammenklänge*, the subject of figured bass. We shall briefly survey the development of *Klangschlüssel* in Riemann's work.

Riemann began to experiment with harmonic symbols in "Ueber das musikalische Hören." These early dabblings, which are reproduced in Example 5-1, led directly to his use of *Klangschlüssel* in *Musikalische Syntaxis*. The notation shown in 5-1a occurs nowhere else in Riemann's work. On its surface it resembles figured-bass notation, however the numbers refer not to intervals but to harmonic overtones (and undertones in one case). Notice that Riemann appends the suffix '^{35}', whether the chord is in root position (chord 5) first inversion (chords 1 and 3), or second inversion (chord 4). In each case, 3—the third partial—stands for the fifth of the chord, and 5—the fifth partial—stands for the third of the chord. The subscripted prefix '$_{53}$' in chord 2 gives notational emphasis to the opposing principle of harmonic undertones. Here 3 stands for the fifth below the root (C), and 5 stands for the third below the root. This prefix would be used no matter what the inversion.

EXAMPLE 5-1: KLANGSCHLÜSSEL PROTOTYPES FROM "ÜBER DAS MUSIKALISCHE HÖREN"



In Example 5-1b, Riemann substitutes the abbreviations '+' and '°' for '^{35}' and '$_{53}$', and invites the reader to compare his use of these symbols with Oettingen's. The difference is small but recalls for us the problematic subject of minor harmony. Oettingen had placed the symbols '+' and '°'

after *Buchstaben*, which was consistent with his use of overtones to explain both major and minor harmony. Riemann follows Oettingen in placing '+' after Buchstaben, but repositions '°' along the lines of '$_{53}$', so that it appears as a subscripted prefix ('$_{\circ}$c' rather than 'c°'.) This repositioning is consistent with his dualism-by-undertones, which stressed the downward nature of minor harmony. A seemingly trivial difference in notation thus underscores an important difference between Riemann's and Oettingen's brands of harmonic dualism.

The harmonic analysis in Example 5-1b requires further decoding, particularly the analysis of sonorities 2 and 3. Riemann views both as embellishments of a consonant g+ triad, and the whole progression as an elaboration of c+ – g+ – c+. Because sonorities 2 and 3 contain four different notes, they are understood as products of more than one triad. Riemann analyzes the second as a product of G–B–D and F–A♭–C, and the third as a product of G–B–D and F–A–C. This notion of triadic simultaneity—of what one might term *Zusammenakkorde*—is the core of Riemann's definition of musical dissonance. To be more precise, Riemann defines dissonance as the coincidence of two or more triads, each of which supports a different cadential *Moment*.[9] A clash of synthetic (G–B–D) and antithetic (F–A♭–C, F–A–C) *Momente* is responsible for the dissonances in Example 5-1b. This clash does not imply functional ambiguity, for one component triad will invariably emerge as the nucleus of a dissonant sonority. G–B–D has this status in the penultimate chord of Example 1b because all three of its pitches are present; the antithetic

[9] Riemann, "Ueber das musikalische Logik," 64: "we saw several voices moving through the different cadential moments simultaneously, and found therein the essense of dissonance." [wir sahen mehrere Stimmen gleichzeitig durch verschiedene Cadenzmomente hindurchgehen und fanden darin das Wesen der Dissonanz.]

F–A–C poses a mild challenge to this triad, since A and C are extrapolations of F and not literally present in the chord. Riemann shows the primacy of G–B–D by placing 'g+' on the same plane as the flanking 'c+'s, and a subordinate 'f' below this. The notation of the antepenultimate takes a similar tack. What is interesting here is that the nucleus is the triad with the fewest sounding pitches; G–B–D is again asserted, but on the strength of D alone, to which Riemann subordinates an entire F minor triad. He writes $(g)^3$ rather than 'g+' to show that G is not literally sounding but is implied by its third partial, the D in the lowest voice; '$_{\circ}c$' is placed beneath $(g)^3$, to show its subordinate status.

Occasionally Riemann's method reveals certain advantages. The Italian sixth in Example 5-1c, for example, is seen as a product of D–F#–A and F–Ab–C, or another clash of synthetic and antithetic *Momente*. The triads supporting these two *Momente* are both incomplete, but the subdominant F minor has more sounding pitches than the applied dominant D major. Riemann nevertheless weights F# over Ab–C, allowing the synthetic *Momente* to prevail on the strength of a single pitch, and writes $(d)^5$ above $_5c$. Superscripted 5 stands for F#, the fifth partial of (implied) D; subscripted 5 stands for Ab, the lower fifth partial of C. Present-day musicians generally favor a subdominant to a dominant reading, but precisely this either-or approach obscures the fact that both tendencies are inherent in the chord.[10] Riemann's weighting of F# ($\sharp\hat{4}$)

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[10] Aldwell and Schachter claim that the chordal background for augmented sixths is "the familiar Phrygian cadence in minor" (p. 479), and that augmented-sixth chords "frequently represent a chromatically inflected IV or II on the way to V" (p. 500); Piston/DeVoto, on the other hand, claims that augmented sixths "have a family resemblance as various types of V of V" (p. 416), citing $\sharp\hat{4}$ as "the clue to the secondary-dominant function" (p. 414); Walter Piston, *Harmony*, 4th ed. revised and expanded by Mark DeVoto (New York: Norton, 1978). Komar acknowledges the functional tension inherent in these chords; see Arthur Komar, *Linear-Derived Harmony* (Cincinatti: Overbird Press, 1992).

over A^b (${}^{b}\hat{6}$) is perhaps old-fashioned, but his *Klangschlüssel* at least register the opposing forces at work here.

The system of *Klangschlüssel* was both expanded and weakened in *Musikalische Syntaxis*. The symbols '+' and '°' served as before, in connection with *Buchstaben*, but arabic numbers now represented intervals above or below a given root.[11] A flood of new symbols augmented these basic ones, but were introduced in such ad hoc fashion that their significance within the larger system floundered.[12] Most were subsequently abandoned in *Skizze* and did not reappear in later work. We take note of them here because they provide an interesting glimpse into Riemann's development of an appropriate chord notation. Example 5-2 consolidates the *Klangschlüssel* of *Musikalische Syntaxis* in one column, and gives *Buchstaben* spellings in another. Some comments appear in a third column.

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[11] The *Unterklang* symbol '°' was superscripted, so that '$\circ c$' became '$^{\circ}c$'. Similarly, c^5 (5 = fifth partial) became c^3 (3 = major third).

[12] Riemann's attempt to draw this system together in a footnote at the beginning of *Musikalische Syntaxis* (p. 9) was inadequate. The number of new symbols, and their unusual character warranted a more thorough treatment in the body of the text.

EXAMPLE 5-2: *KLANGSCHLÜSSEL* FROM *MUSIKALISCHE SYNTAXIS*

Klangschlüssel	Buchstaben	Comments
(1) c^{+7}	c–e–g–b^b	7 = minor seventh above root
(2) 7c	d–f–a^b–c	7 = minor seventh below root
(3) $¢^{+7}$	e–g–b^b	¢ replaces (c)
(4) 7¢	d–f–a^b	¢ replaces (c)
(5) c^+g	c–e–g–b–d	*Klangzweiheit* - two major triads
(6) ĉ+	c–e–g–b	not *Klangzweiheit* because c+ dominates
(7) g⌒t	e–g–b–(d)	⌒t = "°Klang der Terz" (later *Terzwechselklang*)
(8) f°c	b^b–d^b–f–a^b–c	*Klangzweiheit* - two minor triads
(9) °c	d^b–f–a^b–c	not *Klangzweitheit* because °c dominates
(10) T⌒b	(e)–g–b–d	T⌒ = "Durklang der °Terz" (later *Terzwechselklang*)
(11) c/g	g–b–d–f–a^b–c	/ = *Divergenz* between triads: f–a^b–c — g–b–d
(12) ¢/g̸	b–d–f–a^b	same as above with c and g omitted
(13) c/d	a^b–c–d–$f^\#$	French sixth (f–ab–c–d–$f^\#$–a is illegitimate)
(14) Λ (lambda)	———	half step suspension from below
(15) °Λ (lambda)	———	half step suspension from above
(16) Γ (gamma)	———	whole step suspension from below
(17) +Γ (gamma)	———	whole step suspension from above
(18) I, III, V, VII	———	indicates bass if placed below *Klangschlüssel*
		indicates upper voice if placed above *Klangschlüssel*

With the exception of numbers (1) through (4), and allowing for slight modifications there, none of the symbols in Example 5-2 survived *Musikalische Syntaxis.* The concept of *Klangzweiheit,* a dissonance in which neither component triad is nucleus, is unique to this treatise. Notations such as (5) and (8) became superfluous in *Skizze,* where Riemann saw ninth chords as emanating from single roots. Their counter-notations in (6) and (9) were also discarded in that work. The symmetrically related (7) and (10) prefigured Riemann's *Terzwechsel* relation. In (7) the root movement G–B (*Terz*) was accompanied by a change of mode (*Wechsel*), so that B became root of the minor triad E–G–B. In (10) the root movement B–G was accompanied by a change of mode, so that G became root of the major triad G–B–D. Numbers (11) and (12) show the origin of the diminished seventh from dominant and minor subdominant triads. Riemann used the term *Divergenz* for chords consisting of one major and one minor component; sonorities such as (12) were seen to pull in two directions—to "diverge" along modal lines—and were especially unstable. Riemann's symbol for *Divergenz* was a slash placed between two *Buchstaben*. In notations such as ¢/g̸ (12) or c/d (13), the first *Buchstabe* was understood as root of an *Unterklang*, the second as root of an *Oberklang*. Both component triads were not necessarily complete: Riemann understood the pitches F and A to be extraneous to c/d, because the resulting harmony F–A^{b}–C–D–F$^{\#}$–A was illegitimate by traditional standards; c/d was instead a shorthand for the French sixth in C major. It expressed the divergent tendencies of the chord without weighting one cadential *Momente* over the other (compare with Example 5-1c). Finally, numbers (14) – (18), though not *Klangschlüssel* themselves, were used in conjunction with *Klangschlüssel*. Riemann held out hopes for the lambda-gamma suspension symbols, which he had tested on the "kühnste Vorhaltsverkettungen" in the Prelude to *Tristan und Isolde*. Such symbols would typically occur in connection with the roman numerals of (18): °Λ–V written over g^{+7} for a half step suspension resolving downward to the fifth (V) of G, thus E^{b}–D; or Γ–III–Γ–V over d^{+7} for a chain of whole step suspensions resolving upward to the third (III) and then the fifth (V) of D, thus E–F$^{\#}$–G–A.[13] Roman numerals enabled

[13] This last example is from Riemann's analysis of Schubert's Impromptu op. 90, no.3; see *Musikalische Syntaxis*, 67.

Riemann to give detail about the voicing and inversion of a chord. For example, a V written below c+ ($\underset{V}{c+}$) meant that G was in the bass (a six-four chord); a V written above c+ ($\overset{V}{c+}$) meant that G was in an upper voice (was not omitted from the chord). This use of roman numerals was modified in *Skizze*.

Riemann had had almost a decade to reflect on notation by the time of *Skizze*, and was ready to present the definitive version of his "neue Bezifferung." He did so at the beginning of the work, in a section entitled "Klänge, Klangvertretung, Klangschlüssel."[14] The new notation was consistent and easy to learn, but not without disadvantages. Riemann represented major and minor triads as before, but now used both arabic numbers and roman numerals to represent chordal dissonance. Arabic numbers were reserved for notes added to major triads, and roman numerals for notes added to minor triads. Why use both? Since Riemann measured intervals upward from the roots of major triads and downward from the roots of minor triads, he probably felt it was confusing to use arabic numbers for both kinds of measurement—better to have one set of symbols for upward intervals and another set for downward intervals. In the new system, arabic numbers corresponded to notes above a major root, and roman numerals to notes below a minor root (the "root" being the fifth of the minor triad). Taking C as root, we map notes to arabic numbers as follows: C $\rightarrow$ 1, D $\rightarrow$ 2, E $\rightarrow$ 3 ... B $\rightarrow$ 7; and notes to roman numerals: C $\rightarrow$ I, $B^b \rightarrow$ II, $A^b \rightarrow$ III ... $D^b \rightarrow$ VII.[15] Riemann's mapping for the major roots C (c^+), $G^{\#}$ (gis^+), F^b (fes^+) and minor roots E (°e), A^b (°as), $B^{\#}$ (°his) is given below in Example 5-3.

14 Riemann, *Skizze*, 1–17. Riemann presents the *Klangschlüssel* on pp. 10–17, in a subsection entitled "Die neue Bezifferung."

15 Mod. 8 arithmetic applies, so that 1/I = 8/VIII, 2/II = 9/IX, 3/III = 10/X and so on.

EXAMPLE 5-3: ARABIC NUMBERS AND ROMAN NUMERALS IN *SKIZZE*

Die Bedeutung der + Zahlen (Oberzahlen) ist:

	für c^+:	für gis^+:	für fes^+:
1 = Durprim, Hauptton des Oberklangs	*c*	*gis*	*fes*
2 = (Ober-)Sekunde	*d*	*ais*	*ges*
3 = (Ober-)Terz	*e*	*his*	*as*
4 = (Ober-)Quarte	*f*	*cis*	*heses*
5 = (Ober-)Quinte	*g*	*dis*	*ces*
6 = (Ober-)Sexte	*a*	*eis*	*des*
7 = (Ober-)Septime	NB. *b*	*fis*	*eses*

8 = Oktave (= 1; nur als melodische Nebennote der Septime oder None mit 8 beziffert).

9 = None (= 2; nur in Akkorden, die entweder Prim und Terz enthalten, sodass deren Vertretung durch die Sekunde nicht anzunehmen ist, oder aber in Akkorden, die auch die Septime enthalten, mit 9 beziffert).

10 = Dezime (= 3, doch so nicht von Bedeutung; nur die von der 3 verschiedene erniedrigte Dezime als Nebennote der erniedrigten None ist gelegentlich bedeutsam).

Die Bedeutung der ° Zahlen (Unterzahlen) ist entsprechend:

	für ^0e:	für ^0as:	für ^0his:
I = Mollprim, Hauptton des Unterklangs	*e*	*as*	*his*
II = Untersekunde	*d*	*ges*	*ais*
III = Unterterz	*c*	*fes*	*gis*
IV = Unterquarte	*h*	*es*	*fisis*
V = Unterquinte	*a*	*des*	*cis*
VI = Untersexte	*g*	*ces*	*dis*
VII = Unterseptime	NB. *fis*	*b*	*cisis*

VIII = Oktave (= I)
IX = None (= II)
X = Dezime (= III)
} nur ausnahmsweise von I, II, III unterschieden.

Riemann also introduced the symbols '<' and '>', to be used with numbers and numerals in cases of chromatic inflection. One might expect a pitch number associated with '<' to be lower than the same number associated with '>', as in '5<' < '5' < '5>'. Just the opposite was true. Riemann used '<' to denote raising by a semitone and '>' to denote lowering by semitone, so that '5>' < '5' < '5<'. Another level of confusion arose from Riemann's use of '<' and '>' in the *Leittonwechselklänge*. Here intuition was confirmed, in the sense that a function symbol plus '<' (<T, <S, <D) expressed a minor chord, and a function symbol plus '>' (T>, S>, D>) expressed a major chord; the minor third of <T was indeed "less than" the major third of T>.[16] By confirming intuition in one case and running afoul of it in another, Riemann conflicted the sense of '<' and '>' for future students of his theory. The symbols persisted, however, and appeared occasionally in the work of other theorists.[17] Example 5-4 combines '<' and '>' with arabic numbers and roman numerals, and maps these combinations along the lines of Example 5-3. Notice that '$4^{>}$' is equivalent to '3', and therefore "unverständlich" as a dissonance. '$IV^{<}$' is likewise equivalent to 'III' and "undenkbar." Finally, '$7^{>}$' and '$VII^{<}$' refer to doubly flatted sevenths—B^{bb} (heses) above C (c^{+}); $F^{\#\#}$ (fisis) below E (°e)—rather than to minor sevenths. Riemann reserved the symbols '7' and 'VII' for minor sevenths, and '$7^{<}$' and '$VII^{>}$' for major sevenths.

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[16] We remind readers of our convention to write Wechsel signs '<' and '>' beside function symbols rather than through them.

[17] See Alfred Lorenz, *Der Musikalische Aufbau von Richard Wagners "Tristan und Isolde,"* vol. 2 of *Das Geheimis der Form bei Richard Wagner*(Berlin: Max Hesses Verlag, 1926), 3.

EXAMPLE 5-4: THE SYMBOLS '<' AND '>'

Die Bedeutung der Zahlen mit < und > ist nun übersichtlich

	für c^+:	für gis^+:	für fes^+:
1< erhöhte Prim	*cis*	*gisis*	*f*
1> erniedrigte Prim	*ces*	*g*	*feses*
2< erhöhte Sekunde	*dis*	*aisis*	*g*
2> erniedrigte Sekunde	*des*	*a*	*geses*
3< erhöhte Terz	*eis*	*hisis*	*a*
3> erniedrigte Terz	*es*	*h*	*asas*
4< erhöhte Quarte	*fis*	*cisis*	*b*
4> erniedrigte Quarte	(unverständlich)		
5< erhöhte Quinte	*gis*	*disis*	*c*
5> erniedrigte Quinte	*ges*	*d*	*ceses*
6< erhöhte Sexte	*ais*	*eisis*	*d*
6> erniedrigte Sexte	*as*	*e*	*deses*
7< erhöhte Septime	*h*	*fisis*	*es*
7> erniedrigte Septime	*heses*	*f*	${}^{3}\flat_e$

8< = 1<, 8> = 1>,
9< = 2<, 9> = 2>,
10< = 3<, 10> = 3>.

	für ^0e:	für ^0as:	für ^0his
I> erniedrigte Mollprim	*es*	*asas*	*h*
I< erhöhte Mollprim	*eis*	*a*	*hisis*
II> erniedrigte Untersekunde	*des*	*geses*	*a*
II< erhöhte Untersekunde	*dis*	*g*	*aisis*
III> erniedrigte Unterterz	*ces*	*feses*	*g*
III< erhöhte Unterterz	*cis*	*f*	*gisis*
IV> erniedrigte Unterquarte	*h*	[illegible]	*fis*
IV< erhöhte Unterquarte	[illegible]		
V> erniedrigte Unterquinte	*es*	*deses*	*e*
V< erhöhte Unterquinte	*ais*	*d*	[illegible]
VI> erniedrigte Untersexte	*ges*	*ceses*	*d*
VI< erhöhte Untersexte	*gis*	*c*	[illegible]
VII> erniedrigte Unterseptime	*f*	*heses*	*cis*
VII< erhöhte Unterseptime	*fisis*	*h*	[illegible]

VIII> = I>, VIII< = I<,
IX> = II>, IX< = II<,
X> = III>, X< = III<.

Whereas the older *Klangschlüssel* emphasized the components of chords, the new *Klangschlüssel* emphasized chordal roots: One now wrote $\not{g}^{9>}$ instead of $\not{c}/\not{g}$, to show that G alone was root of the chord b–d–f–a^b. For the chord d–f–a–c, one wrote either $^{\circ}a^{VI}$ or f^{+6} depending on whether one meant the supertonic seventh (A as root) or the subdominant "chord-of-added-sixth" (F as root). Root specificity was not necessarily a step forward for Riemann's notation. A nice feature of the earlier *Klangschlüssel* was their openness to harmonically ambiguous or divergent chords. The idea behind pairs of symbols such as $^{\circ}a^{VI}$ and f^{+6}

was to resolve ambiguity, but this was no advantage for chords that were inherently two-sided. The French sixth a^{b}–c–d–$f^{\#}$, for example, was better served by c/d than by $ab^{6<}$, since the older notation analyzed the chord into antithetic and synthetic *Momente* while the newer merely described its content. It is not clear to what extent the *Klangschlüssel* of *Skizze* signaled a shift in Riemann's thinking about harmony. The new notation was perhaps ancillary to the earlier move from musical logic to musical syntax, but the real motive for change was probably practical. Riemann was anxious for *Skizze* to succeed, and he simplified his notation (among other things) accordingly. The triad remained the basis of his system, but *Klangschlüssel* now itemized chord content instead of analyzing it. Ironically, in this new and definitive version, the *Klangschlüssel* were closer to the figured-bass notation that Riemann had set out to reform: c^{+6} (not $c \frown t$) = a–c–e–g; $c^{+6<}$ = $a^{\#}$–c–e–g; $c^{+6>}$ = a^{b}–c–e–g, and so on. Implied notes were struck as before, $\not{g}^{+7}$ for b–d–f, whereas chord voicing and inversion were shown by means of arabic and roman numerals: $\overset{3}{c}$ = c–e; $\underset{5}{c}$ = g–c; $\underset{\text{III}}{e}$ = c–e; $\overset{\text{V}}{e}$ = e–a.

5.5 *Functionbezeichnung*

Riemann's function symbols are better known than his *Klangschlüssel*, and we shall spend less time in our survey of them. Tonic, subdominant, and dominant are symbolized T, S, and D in major and °T, °S, and °D in minor. We remind the reader that Riemann associated these symbols with specific chords, rather than chordal categories, which led to awkward formulations where, for instance, chords other than T were said to be tonic in function. The chords symbolized by T, S, and D, and their minor counterparts, must be understood as prototypes of Riemann's three harmonic functions. All other chords are either direct transformations of these prototypes, or second-order transformations of transformations. In both cases, the result of transformation is symbolized through a combination of T/°T, S/°S, or D/°D—the basic harmonic functions—with other symbols indicating the relation between transformed chord and prototype. The *Parallel* and *Leittonwechsel* are the most important of these relations, and use the symbols 'p' and '<' or '>', respectively. Riemann's *Parallel* relation is equivalent to the relative major-minor relation in American music theory: If the tonic T is C–E–G, then the *Parallel* Tp is A–C–E; if the subdominant S is F–A–C, then the *Parallel* Sp is D–F–A; if the dominant D is G–B–D, then the *Parallel* Dp is E–G–B. The term *Parallel* is in some ways more apt than the term "relative." If T and Tp are thought of as key areas rather than chords, one sees that the respective key signatures are not merely related but identical, and that for any pitch in T there is an analogue, or parallel pitch, whose tonal meaning is the same in Tp. The spatial connotations of the parallel relation are made vivid by the Table. If we take T to represent the tonic of C major and Tp to represent the tonic of A minor, then the horizontal fifth series that stem from C (tonic of T) and E (tonic of Tp) form two parallel lines. If we take T and Tp to represent chords, we see that they are not only adjacent on the Table but form a parallelogram.[18] The same is obviously true of S and Sp, and D and Dp. Example 5-5 summarizes the parallel relation in tabular format.

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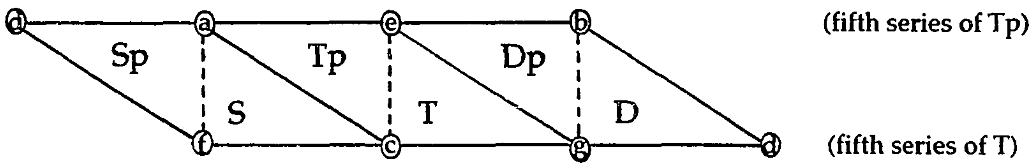
[18] Imig, 14.

EXAMPLE 5-5: THE *PARALLEL* RELATION



By means of T, S, D and Tp, Sp, Dp, Riemann accounts for the three major and minor triads of any given major key (°T, °S, °D, and °Tp, °Sp, °Dp are the minor counterparts). The parallel relation is insufficient, however, since the chord symbolized by Tp does not always function as a tonic, and the chord symbolized by Dp does not always function as a dominant. Dp has two tones in common with T and two in common with D (see Ex. 5-5), and will thus function as tonic in some contexts and dominant in others. Tp likewise shares tones with T and S and will function as tonic in some contexts and subdominant in others. Riemann's *Leittonwechsel* relation provided for alternative interpretations of the mediant (Dp) and submediant (Tp) triads.

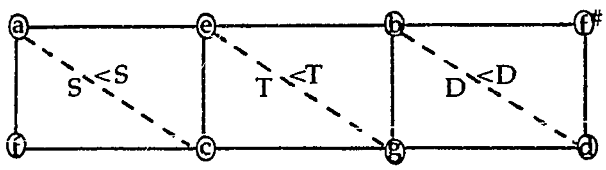
The *Leittonwechsel* relation was not Riemann's invention, but an artifact from Oettingen's theory of harmonic dualism. While Riemann and Oettingen both recognized the *Leittonwechsel* relation, only Riemann ascribed functional significance to chords that resulted through its application (a point we shall return to presently). The *Leittonwechsel* is just one of several *Wechsel*-type relations in Riemann's theory, and may thus serve as a primer to the SW-system in general. Earlier we spoke of *Schritte* and *Wechsel* as the two kinds of transformation that Riemann uses, and said that the phrase "Systematik der Harmonieschritte" was only half accurate. The term *Leittonwechsel* is also only half accurate, since this transformation involves both a *Schritt* and *Wechsel*, a *Leittonschritt* to be exact, followed by what Riemann called the *Seitenwechsel*. *Leittonwechsel* is thus an ellipsis of "Leittonschrittseitenwechsel," and the transformation works as follows: Starting with T = C–E–G, one derives the major triad a major seventh above C. This is the *Leittonschritt*. One then inverts the new triad B–$D^{\#}$–$F^{\#}$ around its root. This is the *Seitenwechsel*. The result is the minor triad E–G–B, which we formerly symbolized as Dp but which we symbolize as <T under the *Leittonwechsel* relation. If we begin with a minor chord, the *Leittonwechsel* yields a major chord: If °T = A–C–E, one first derives the minor chord whose root lies a major seventh below E. One then inverts this triad B^{b}–D^{b}–F around its root to get the major triad F–A–C. Under the *Leittonwechsel* relation, the submediant becomes T> rather than °Sp. As we noted above, <T, <S, and <D are always minor, and T>, S>, and D> are always major. It is perhaps easiest to grasp the net effect of the *Leittonwechsel* in terms of the Table. Just as *Parallel*-related chords form parallelograms, *Leittonwechsel*-related chords form squares.[19] Example 5-6 summarizes the the *Leittonwechsel* relation in major and minor, and should be compared with Example 5-5. Among <T, <S, and <D in major, <T is essential since the mediant triad normally functions as part of a tonic expansion (<T rather than Dp) leading to the dominant. The submediant triad may serve as a tonic substitute (Tp), or occur in bass arpeggiations expanding tonic (T–Tp–S) or subdominant harmony (T–<S–S). <D is a chromatic chord that is rarely used. In minor, S> symbolizes the Neapolitan (^{b}II) chord,

[19] Imig, 15.

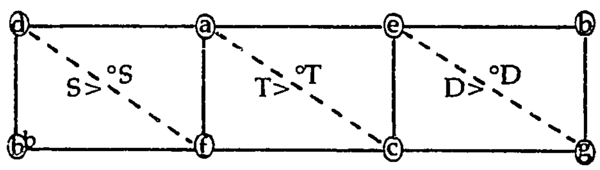
and is invariably subdominant in function; T> functions analogously to the submediant Tp in major (sometimes as tonic, sometimes as dominant); and D> is normally heard as a tonic (°Tp rather than D>).

EXAMPLE 5-6: THE *LEITTONWECHSEL* RELATION

C major



A minor



The *Parallel* and *Leittonwechsel* transformations, along with T, S, D and °T, °S, °D, exhaust the diatonic triads (and account for some chromatic ones) in any given major or minor system. Chromatic chords can be derived through other transformations, but no additional symbols are needed to express the results of these transformations. The basic set of function symbols consists of just nine elements in major {T, S, D, Tp, Sp, Dp, <T, <S, <D} and nine in minor {°T, °S, °D, °Tp, °Sp, °Dp, T>, S>, D>}.

Riemann uses function symbols in two ways to represent chromatic chords: 1) he symbolizes the chromatic chord as an applied subdominant or dominant of one of the more basic chords. Simple examples would be the dominant-of-the-dominant, written 'D^{D}', or the subdominant-of-the-subdominant, written '${}^{S}{}_{S}$'. More complex examples would be (D)$^{\circ}{}_{D}{}^{D}$ and (°S)S>, the first indicating the "major dominant of the minor dominant of the minor dominant" ($F^{\#}$–$A^{\#}$–$C^{\#}$ in A minor), the second indicating the "minor subdominant of the *Leittonwechselklang* of the minor subdominant" (G^{b}–B^{bb}–D^{b} in C major); 2) Riemann also symbolizes chromatic chords by means of "hybrid" notation. This notation consists of a function symbol, which expresses the basic chord, and *Klangschlüssel* "figures," which alter this chord as needed. For example, if Sp in C major—a D minor triad—were followed immediately by a D major triad, Riemann might use the symbol $Sp^{III<}$, to show that $Sp^{III<}$ is merely an inflection of Sp, and that subdominant functionality prevails. Hybrid notation affords Riemann some interpretive leeway by enabling distinctions along the lines of those observed in connection with the horizontal *Striche*. $Sp^{III<}$ is on the subdominant side of T and contains the pitches $\underline{d}$–$\underline{\underline{f}}^{\#}$–$\underline{a}$, whereas ${}_{D}{}^{D}$ is on the dominant side and contains the pitches d–$\underline{f}^{\#}$–a. The difference between $Sp^{III<}$ and ${}_{D}{}^{D}$, in C major, is thus analogous to the difference between $\underline{d}$–$\underline{\underline{f}}^{\#}$–$\underline{a}$ and d–$\underline{f}^{\#}$–a. Without recourse to hybrid notation, it would be awkward indeed to express the contrasting functions of chords such as $Sp^{III<}$ and ${}_{D}{}^{D}$, whose pitches share the same letter names but occupy different positions on the Table.

Most of the symbols that appear in Imig's *Tonnetz* should be decipherable at this point, and readers may want to reacquaint themselves with Example 3-1 in Chapter 3 of the present study. One symbol that we have not discussed but that Imig includes is 'v', representing the so-called *Variante* relation. Riemann introduced this late in his career, perhaps to replace *Klangschlüssel* in hybrid forms such as $\mathrm{Sp}^{III<}$.[20] The *Variante* changes major triads into their minor counterparts and vice versa, as in the Sp to $\mathrm{Sp}^{III<}$ example we discussed above. One could write Sp^{v} for $\mathrm{Sp}^{III<}$ in that example, showing that D major is simply the inflected variant of Sp. There is nothing new in the *Variante* concept, but the symbol 'v' did help to unclutter Riemann's notation. This is significant, for the whole phenomenon of hybrid notation—notwithstanding certain advantages—underscores the basic hermenuetic problem of Riemann's function symbols: What exactly do they mean?

We have crossed the hurdle from category to chord in our discussion of harmonic function, and shall not recross it here. In Chapter 3 we noted that T, S, and D serve at cross purposes, as labels for harmonic functions as well as specific chords. We must now consider the possibility that *Parallel* and *Leittonwechsel* symbols stand not just for functions or chords, but also for chordal transformations. It is not clear from the symbols alone whether 'p' in Tp, or '<' in <T, expresses a product (chord) or a process (transformation). This apparent division of labor is only complicated by the introduction of hybrid notation. Combinations such as $\mathrm{Sp}^{III<}$ seem to package function, transformation, and chord description into one all-purpose symbol. For the remainder of this study, we shall ignore the transformational implications of Riemann's symbols and

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[20] The *Variante* was introduced in *Handbuch der Harmonielehre*, 6th ed., (Leipzig, 1918). In the foreword, Riemann called the symbol 'v' a "Hilfszeichen für Ausnahmserscheinungen," and cited the minor Neapolitan at the end of Schubert's Impromptu op. 90, no. 3 as a case where he would write $\mathrm{S}{>}^{v}$ instead of $\mathrm{S}{>}^{3>}$. Riemann also defined *Variante* in his *Musiklexicon* (Berlin and Leipzig, 1916) as the "term that identifies the major form of the minor tonic or minor form of the major tonic, which is substituted through the alternation of the third (major instead of minor, minor instead of major)." [Terminus, der die durch Veränderung der Terz (groß statt klein, klein statt groß) substituierte Durform der Tonika der Molltonart oder die Mollform derjenigen der Durtonart kennzeichnet.]

associate them—as Imig does in the *Tonnetz*—with functionally-invested chords. In what follows, the focus is on neither chords nor function symbols, but on the *Schritt-Wechsel* processes through which chords are derived.

5.6 Introduction to the SW-System

Whereas the theory of harmonic functions emphasized the meaning of chords, the system of *Schritte* and *Wechsel* emphasized the motion between chords. The terms *Schritt* and *Wechsel* were great improvements on Riemann's older terminology, obviously because they were less arcane, but also because they imparted a kinetic sense to his concept of harmony. Several of the relationships we discuss below were identified in Chapter 3, and readers should review the summary of "one-sided thesis" given in Example 3-19 of that chapter. By grouping the major and minor counterparts in Example 3-19, we can reduce Riemann's harmonic syntax to just ten moves. By comparison, there are twenty-five different *Schritte* and *Wechsel* in *Skizze* and thirty-two in *Handbuch*. Riemann's sense of what counted as an intelligible chord progression clearly broadened over time.[21] Of the total number of *Schritte* and *Wechsel*, twenty-five were integral to the SW-system and seven—the ones added in *Handbuch*—were restricted "nur in einzeln Formen." Apart from these more specialized relationships, there was only one notable difference between

[21] Again it is tantalizing to consider the influence of contemporary practice. The 6th ed. of *Handbuch* appeared near the end of a decade that had brought forth such works as *Le sacre du printemps* (1913) and *Pierrot lunaire* (1912).

the presentation of *Schritte* and *Wechsel* in *Skizze* and in *Handbuch*: Riemann applied chord transformations to *Klangschlüssel* in the earlier work, and to function symbols in the later work. Since the presentation is otherwise so similar (there is some reordering of material),[22] we shall use *Handbuch* as our reference and note any discrepancies with *Skizze* as we proceed.

5.7 *Schritte* and *Wechsel*

Schritte relate chords of the same mode, and *Wechsel* relate chords of opposite mode. One can describe these relationships in different ways, but music theorists would tend to describe them in terms of transposition and inversion. Though *Schritte* are indeed the result of transposition, and *Wechsel* the result of transposition followed by inversion, we shall use terms other than these in much of our discussion. For the time being it is important only to realize that *Schritte* are simpler than *Wechsel*. They are in a sense "single action" transformations, involving transposition alone, whereas *Wechsel* are "double action." We have touched upon this difference in our discussion of the *Leittonwechsel*, which we said was shorthand for "Leittonschrittseitenwechsel," but should stress that *Wechsel* was Riemann's shorthand for any *Wechsel* relation. The obvious exception to this rule is the *Seitenwechsel*—or actual change-of-mode—which indicates inversion but not transposition.

[22] In *Skizze*, numerous part-writing *Aufgaben* are grouped in an appendix credited to Adolph Weidig. These exercises are interspersed throughout the text of *Handbuch*, with no mention of Weidig. Riemann also incorporates exercises into *Handbuch* that employ his new function symbols.

If one thinks of transposition and inversion more dynamically, as motion "toward" in the first case, and motion "around" in the second, one comes closer to Riemann's sense of harmonic *Schritte* and *Wechsel*. *Schritte* are motions toward pitches, chords, or key areas; the concept is hierarchical. Normally, Riemann treats just chordal *Schritte*—pitches are harmonic for him in any case—but in *Skizze* he treats key as well as chord. The harmonic motion described by *Schritte* is either homologic or antilogic; however Riemann no longer uses these terms. Instead, a motion in the same direction (homologic) as the governing harmonic series is a *schlichter* (plain) *Schritt*, and a motion in the opposite direction (antilogic) is a *gegen* (contra) *Schritt*. *Schritte* are directed motions, in other words, and come in the following sizes: *Quint* (fifth), *Terz* (major third), *Kleinterz* (minor third), *Ganzton* (major second), *Halbton* (minor second), and *Tritonus* (tritone). A *schlichter Quintschritt*, "der leichtest verständliche" among *Schritte*, would describe the motion between c+ and g+ (T and D in C major). A *gegen Quintschritt*, on the other hand (whose cadential effect is "nur eine unvollkommene") would describe the motion between c+ and f+ (T and S in C major). Note that ordering makes a difference in these examples. The motion g+ – c+ is a *gegen Quintschritt*, because c+ is the lower fifth of g+, whereas f+ – c+ is a *schlichter Quintschritt*.

Wechsel relations are also motions "toward" but are more essentially motions "around" a harmonic root. There is some question as to how essential this second kind of motion was for Riemann. Normally he treated change-of-mode without particular bias; however *Handbuch* and *Skizze* both contained overviews of the "Systematik der Harmonieschritte" that emphasized root progression over modal change.

Because *Wechsel* relations did involve root progression, Riemann classified them on this basis and grouped them with genuine *Schritte,* into chord transformational families. Each of these families contained four members—*schlichte* and *gegen Schritte,* plus their *Wechsel* counterparts—and membership was based on the size (*Quint, Terz,* and so on) rather than direction (*schlicht* or *gegen*) of a given *Schritt.* The family of *Quintschritte* thus consisted of the *schlichter Quintschritt* (T–D; °T–°S), the *Gegenquintschritt* (T–S; °T–°D), the *schlichter Quintwechsel* (T–(°S)D; °T–(D)°S), and the *Gegenquintwechsel* (T–(°S)°S; °T(D)D). Such organization suggests that *Wechsel* relations were a species of *Schritt* for Riemann, subsumed within his "Systematik der Harmonieschritte." We shall develop our notion of a chord transformational family below.

5.8 SW-sets

Six chord-transformational families, each with four members, account for twenty-four *Schritte* and *Wechsel.* The *Seitenwechsel* comprises a family unto itself—a singleton in set theoretical terms—and brings this number up to twenty-five. To be consistent with our notion of an SW-system, we shall call these families SW-sets; the SW-system thus consists of six SW-sets. These sets take their names from the different *Schritt* sizes, which roughly determine the order of presentation in Riemann's "Systematik der Harmonieschritte": *Quintschritte* come first, followed by *Terzschritte, Kleinterzschritte, Ganztonschritte, Halbtonschritte,* and finally *Tritonusschritte.*[23] As intervallic consonance declines, so too does

[23] The "Systematik der Harmonieschritte" is presented on pp. 124–35 of *Handbuch* and pp. 113–26 of *Skizze.*

the intelligibility of harmonic transformation. This at any rate is the organizing principle. Often a member of an SW-set will describe a more remote relationship than members belonging to sets that Riemann presents later. Among *Terzschritte,* for example, the *Gegenterzwechsel* expresses (in C major) a relatively obscure motion from c+ to $^{\circ}a^{b}$, or tonic (T) to minor Neapolitan ($S>^{3>}$ or $S>^{v}$). This latter chord occurs at the end of Schubert's Impromptu in G^{b}, op. 90, no. 3 in an expansive cadential gesture, which we cite in Example 5-7. Riemann's *Klangschlüssel* (transposed into G^{b} major) are given immediately beneath the excerpt, and his function symbols below these. Notice that $S>^{3>}$ is none other than the antinomic-antilogic *Terzklang,* a term that Riemann had used in his analysis of this piece in *Musikalische Syntaxis.* $S>^{3>}$ is a rare chord under any name, and a measure of its intelligibility is found in *Skizze*—in a discussion of key relationship omitted from *Handbuch*—where Riemann ranks the *Gegenterzwechsel* ninth out of twelve possible degrees of relation. Only the *Gegenkleinterzwechsel* ($^{\circ}e^{b}$), *Tritonuswechsel* ($^{\circ}f^{\#}$), and *Doppelterzwechsel* ($^{\circ}g^{\#}$) are more remote.[24]

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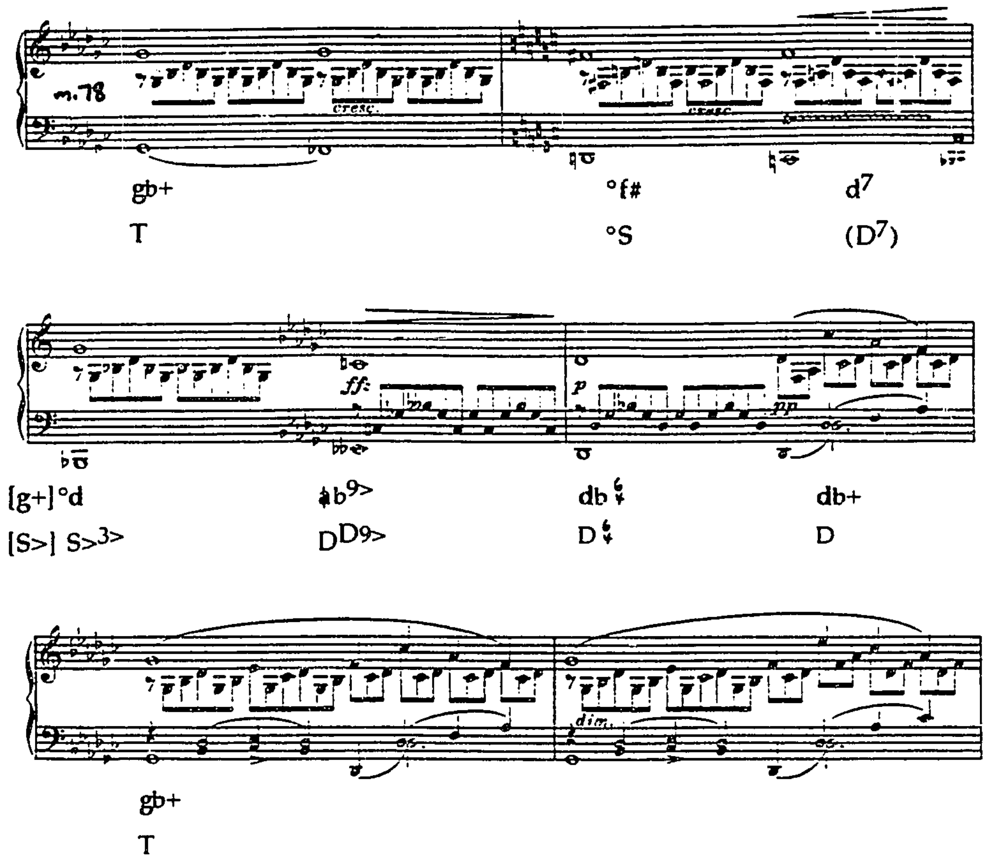
[24] Riemann, *Skizze,* 133–35. Riemann divides the twelve degrees of key relation into two groups: primary (*erste Linie*) keys, and second-order (*zweite Ordnung*) keys. The primary relations to C major are: 1) F minor, 2a) G major, 2b) F major, 2c) C minor, 3a) E major, 3b) A^{b} major, 3c) A minor , 4a) A major, 4b) E^{b} major, 4c) D minor, and 5) E minor. The second-order relations are: 6a) D major, 6b) B^{b} major, 6d) G minor, 6e) E^{b} minor, 7) B^{b} minor, 8a) D^{b} major, 8b) B major, 8c) G^{b} minor, 9) D^{b} minor, 10) A^{b} minor, 11) B minor, and 12) $C^{\#}$ minor. This ordering is at odds with the "Modulationstäfelchen" in *Systematische Modulationslehre* (also published in 1887), which lists B major, B minor, G minor, and D^{b} major as direct relations, but excludes F minor, A minor, and D minor. See n. 45.

EXAMPLE 5-7: THE *GEGENTERZWECHSEL* IN SCHUBERT'S IMPROMPTU, OP. 90, NO. 3



The *Schritt* sizes in Riemann's SW-system comprise all six interval classes. This means that root progression between any two pitches of the chromatic scale is possible, whereas in *Musikalische Syntaxis* only progressions by perfect fifth and major third were permitted.[25] Riemann's earlier system was versatile, but ruled out direct motions to nondiatonic chords. The *Tritonusschritt*, for example, could be expressed as 2Q + T—two perfect fifths followed by a major third—but had no independence as a

[25] The *Leittonwechsel* was explained as an inversion around the sixth G–E in C major, or C–A in A minor, and not as a leading-tone progression. For this reason the chord was called a *Sextenwechselklang* (see 3.13).

root progression consisting of three whole-steps. Now it did; and other progressions that had been mediated through Q and T were simultaneously reborn. The virtue of the SW-system was not sudden or permanent liberation from Q and T but greater flexibility. In most cases it made good sense to think in terms of Q and T; occasionally, however, it made better sense to assert a direct, if slightly more provocative relation between distant chords. Riemann's SW-system gave theorists this option and thereby afforded them a more sensitive tool for harmonic analysis.

The number of *Schritte* and *Wechsel* along with the unusual array of symbols can be daunting to newcomers to Riemann's theory. To see how these diverse elements work together, we must first simplify some technical aspects of the theory. We need in particular a shorthand for the various SW-sets and their members, and a means of representing the application of *Schritte* and *Wechsel*—or combinations thereof—to specific chords. The methodology we use in the remainder of this chapter owes a good deal to Lewin (1987), Hyer (1989), and Klumpenhouwer (1994), as well as to Imig's excellent survey of Riemann and post-Riemannian function theory.[26]

One needs to be clear about what counts as "process" and what counts as "product" in Riemann's theory. Riemann confuses us in this matter, not only with his function symbols, but with terms such as *Quintwechselklang*, which blur the distinction between chord and transformation. What counts as process are the *Schritte* and *Wechsel*

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[26] Brian Hyer, "Tonal Intuitions in "Tristan und Isolde," Ph.D. diss., Yale University, 1989; Henry Klumpenhouwer, "Some Remarks on the Use of Riemann Transformations," *Music Theory Online* 0/9 (1994).

comprising the six SW-sets, and the *Seitenwechsel*. Nothing else. What count as products are generally chords, though key, chord, and pitch are embedded concepts for Riemann. Function symbols and *Klangschlüssel* should be understood primarily as chord labels, and never as indicators of *Schritte* or *Wechsel*. Finally, one should regard the theory of harmonic functions and the SW-system as complementary halves of Riemann's theory; one concerns itself with chords, the other with motions between chords.

5.9 Formalization

Our formalization of Riemann's SW-system is summarized in two appendices at the end of this chapter; we shall work through the main points in this and the following sections, and refer to the appendices whenever appropriate.

We follow Riemann's example and let interval size determine the order in which we present the SW-sets and the *Seitenwechsel*. Since the *Seitenwechsel* is the only transformation that involves no root progression, Riemann presents it first. There is a simpler operation, however, which we include for reasons that will become apparent. This is the so-called Identity operation, or operation that transforms a chord into itself. As in the case of the *Seitenwechsel*, the Identity operation is the sole member of its set. Our abbreviations for the Identity and *Seitenwechsel* relations are I and W, respectively. We have tried to choose a shorthand for SW-sets that is both intuitive and consistent from set to set. Our solution is fairly straightforward: We abbreviate the set of *Quintschritte* as Q4. 'Q' identifies the root progression common to the elements of the set;

'4' identifies the number of elements, and insures that 'Q4' will not itself be mistaken for an element. The elements of Q4 are Q (*schlichter Quintschritt*), -Q (*gegen Quintschritt*), QW (*Quintwechsel*), and -QW (*gegen Quintwechsel*), which we can express compactly as Q4 = {Q, -Q, QW, -QW}. Following this example, we abbreviate the *Terzschritte* as T4 = {T, -T, TW, -TW}; the *Kleinterzschritte* as K4 = {K, -K, KW, -KW}, and so on. Only two details require comment. Riemann classifies *Leitton* relations as *Halbtonschritte*, but we have used L4 instead of H4 for this set in deference to Riemann's own *Leittonwechsel* terminology. Thus, L4 = {L, -L, LW, -LW}.[27] Finally, we have used Z4 for the set of *Tritonusschritte*. Notice that Z and -Z describe motions to enharmonically related chords, as do ZW and -ZW.[28] Because such chords occupy different places on the Table, and are the products of qualitatively distinct moves, we include all four elements {Z, -Z, ZW, -ZW} in Z4.

We use neither function symbols nor *Klangschlüssel* in our formalization, but an "ordered-pair" notation similar to the notation found in Lewin (1987).[29] This notation indicates *Ober-* and *Unter-Klänge*, and consists of two components: (x, sign), where "x" is a harmonic root (C, F#, A♭, etc.), and "sign" takes the value +, -, or ±, depending on whether the sonority is an *Oberklang* (+), *Unterklang* (-), or *Klang* (±). We regard +

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[27] Instead of *schlichte* and *gegen Halbtonschritte*, Riemann presents *steigende* (rising) and *fallende* (falling) *Halbtonschritte*. This departure creates some confusion; the rising progression C–D♭ is equivalent to -L in major, but L in minor, and the falling progression C–B is equivalent to L in major but -L in minor (see app. 1).

[28] Our notation of *Schritte* and *Wechsel* is similar to that used by Klumpenhouwer (1994). The differences are as follows: Klumpenhouwer does not partition Riemann's transformations into SW-sets, and thus has no need for Q4, T4 , K4, G4, L4, and Z4; he also uses R for the *Tritonusschritt* and RW for the *Tritonuswechsel* (and excludes their inverses -R and -RW). We see no reason to contradict Lewin's and Hyer's use of R for the "relative" relation (equivalent to Klumpenhouwer's and our TW), and thus use Z to symbolize tritone relations.

[29] Lewin, 176.

and - as signs of polarity rather than of mode; (C, +) indicates the *Oberklang*, or positive side of C, whereas (C, -) indicates the *Unterklang* or negative side. Lewin interprets + and - as signs for major and minor, and writes (C, -) where we would write (G, -), and (F, -) where would we write (C, -). Our method is less intuitive, but truer to Riemann's dualist outlook. By way of summary, (C, +) = C-*Oberklang* or C–E–G; (C, -) = C-*Unterklang* or F–A^b–C; and (C, ±) = C-*Klang* or F–A^b–C–E–G.[30]

It is a simple matter of combining SW-set notation with ordered-pair notation to symbolize the motion between chords. The application of I to (C,+), for example, results in (C, +). We represent this process by placing SW symbols between ordered pairs, as follows: (C, +) <—I—> (C, +). Because the identity relation holds whether the notation is read left-to-right or right-to-left, we write arrows that point toward both ordered pairs. Whenever a chord transformation is two-way in this sense, we write arrows that point in both directions. Whenever a transformation is one-way, we write arrows that point in one direction, the direction of the terminal ordered pair: (C, +) —Q—> (G, +) or (C, +) <—Q— (G, +). In addition to the identity relation, all *Wechsel* relations are two-way, whereas all *Schritte* are one-way. Example 5-8 summarizes our SW-set notation, and coordinates this notation with ordered pairs.

[30] Lewin uses *Klang* in the sense of "triad"—major or minor—whereas we mean "Unter- and Ober- Klänge of a given root." We have added the symbol ± to express Riemann's concept of *Divergenz*: (C, +) = c+; (C, -) = °c; and (C,±) = c/c

EXAMPLE 5-8: SW-SET NOTATION AND ORDERED PAIRS

I = Identity	(C, +) <—I—> (C, +)
W = *Wechsel*	(C, +) <—W—> (C, –)
Q4 = {Q, -Q, QW, -QW}	(C, +) —Q—> (G, +); (C, +) —¯Q—> (F, +)
	(C, +) <—QW—> (G, –); (C, +) <—¯QW—> (F, –)
T4 = {T, -T, TW, -TW}	(C, +) —T—> (E, +); (C, +) —¯T—> (A^b, +)
	(C, +) <—TW—> (E, –); (C, +) <—¯TW—> (A^b, –)
K4 = {K, -K, KW, -KW}	(C, +) —K—> (A, +); (C, +) —¯K—> (E^b, +)
	(C, +) <—KW—> (A, –); (C, +) <—¯KW—> (E^b, –)
G4 = {G, -G, GW, -GW}	(C, +) —G—> (D, +); (C, +) —¯G—> (B^b, +)
	(C, +) <—GW—> (D, –); (C, +) <—¯GW—> (B^b, –)
L4 = {L, -L, LW, -LW}	(C, +) —L—> (B, +); (C, +) —¯L—> (D^b, +)
	(C, +) <—LW—> (B, –); (C, +) <—¯LW—> (D^b, –)
Z4 = {Z, -Z, ZW, -ZW}	(C, +) —Z—> ($F^{\#}$, +); (C, +) —¯Z—> (G^b, +)
	(C, +) <—ZW—> ($F^{\#}$, –); (C, +) <—¯ZW—> (G^b, –)

The reader should here refer to appendix 1, which maps the entire SW-system to the Table of Relations—in major and minor—and correlates the shorthand notation we have been using with function symbols from *Handbuch* and *Klangschlüssel* from *Skizze*. Section 8 of this appendix, entitled "Other Transformations," includes seven enharmonic variants of SW-set members. All except one of these transformations are marked with an asterisk, to show that they appear in *Handbuch* but not *Skizze*. The unmarked progression, Riemann's *Doppelterzwechsel* (no. 26), is similar to Lewin's SLIDE relation, in which the third of a triad remains stationary while root and fifth "slide" by semitone to change the mode.[31] (in our notation: (C, +) <—SLIDE—> ($G^{\#}$, -)). One relation that is not included in *Skizze*, but that belongs to the SW-system, is the *Gegentritonuswechsel* (no. 25). Since its inverse ZW does appear in *Skizze* (no. 24), and since ZW and -ZW describe motions to enharmonically related chords, Riemann may have felt it unnecessary to

[31] Lewin, 178.

include both. If this was the case, then his decision to add -ZW in *Handbuch* strengthens the claim that there is a functional difference between these inversely related motions.

The advantage of using ordered pairs instead of function symbols may not be obvious to readers. Why not °T—Q—>°S instead of (E, -) —Q—> (A, -), or T —LW—> Dp instead of (C, +) —LW—> (B, -)? The first method is more abstract than the one we use, and does not convey as palpable a sense of harmonic motion. SW-symbols such as Q and LW describe general motions, and need to be anchored to specific sonorites for us to grasp their effect. Another reason for avoiding function symbols is their resemblance to SW-symbols. If T —LW—> Dp seems vague, then T—TW—>Tp is virtually meaningless. We could solve both problems by using *Klangschlüssel*, however, we believe that the signs +, -, and ± are more suggestive than Riemann's +, °, and / (the *Divergenz* symbol).

5.10 Group Structure

The SW-system, as we have developed it, consists of eight sets. We could treat these sets as elements, and characterize the system itself as a superset: SW-system = {{I}, {W}, {Q4}, {T4}, {K4}, {G4}, {L4}, {Z4}}. The operands of this system are the twelve *Oberklänge* and twelve *Unterklänge*. These also comprise a set, which we shall call SPACE, in reference to the spatial-geometrical nature of chords in Riemann's Table. SPACE is the set of all (x, +) or (x, -), such that *x* is a harmonic root (C, D^b, ... B); '+' is an *Oberklang*; '-' is an *Unterklang*.[32] The members of the eight SW-sets are thus a set of transformations on SPACE. In the last section,

[32] SPACE = {(x, +), (x, -) | x = harmonic root, + = *Oberklang*, - = *Unterklang*}.

we applied single transformations to chords, and represented the process using shorthand notation. In this section, we shall study compound transformations and show how the SW-system is a special type of system called a "group." Our focus is not the relationship between transformations and chords, but the relationship among transformations. The structure of this relationship—the mode of interlocking among SW-set members—is the essence of Riemann's harmonic theory. To help us uncover this structure, we shall put our shorthand notation to work and enlist a modified version of the Table that will allow us to plot chord transformations independently of specific chords.

We have not said much about the systematic aspect of the SW-system; we shall construe this system first of all as an "operational system," and then introduce the more limited notion of a mathematical group. An operational system is a set of elements S plus an operation * that satisfies the following condition; for every element a and b of S there exists an element c of S such that $a * b = c$. It is not hard to find examples that meet this condition. The set of positive integers is operational under addition, since the sum of any two of its members is always a positive integer. The same set is not operational under subtraction, however, since the difference between any two of its members is not always a positive integer. We can use ordered pair notation to name a set and its operator: (P, +) names the set of positive integers under addition; (P, –) names the set of positive integers under subtraction. In the first case, addition is said to be "closed" on P, because adding two elements never gives a value outside of P. In the second case, subtraction is said to be "not closed" on P, because subtracting two elements sometimes does give a value outside of P. In both cases, the operator is called a binary operator because it relates just two elements at a time. Finally, the relation of elements through a binary operator is termed "composition"; *a* * *b* is short for "the composition of elements *a* and *b*".

We have no trouble with the concepts of addition and subtraction as they relate to integers, or to numbers in general; in fact, we have a rather sure sense of what we are doing when we place '+' between two numbers, and how it differs from placing '–' between the same numbers. Our sense of the operator is considerably less clear when we compose chord transformations. In some cases it may seem as though we are "adding"; if we follow Q with T, for example, it does not seem unreasonable to write (as we have been doing) Q + T = L. In other cases, addition seems the wrong metaphor. Are we adding or subtracting, or either, when we follow QW with Q? Because of this uncertainly, we shall use a generic symbol * to symbolize the composition of two chord transformations. Expressions such as 'Q * T = L' mean approximately "the net effect of Q followed by T is L." We needn't be more specific than this, for we are ultimately interested in abstracting relationships from the SW-system, not in solidifying our notion of the operator. Rather than writing out each possible composition of *Schritte* and *Wechsel* (including the Identity relation), it is easier to summarize the possibilities in a composition table. Example 5-9 presents such a table for Q4, the family of *Quintschritte*, along with the Identity relation. The elements of Q4 are listed with I, in the left-hand column and top row. The operator * appears in the upper left-hand corner. Using this table to compose QW and Q, one would first find QW in the left-hand column and Q in the top row; one would then read across from QW and down from Q, to find their intersection at W. In this manner, the table shows us that QW * Q = W. Notice that Q * QW = GW (is not the same as

QW * Q = W), and that the composition of inverse elements (Q and -Q; QW and -QW) results in I. The tables in Example 5-9 express composition abstractly, using SW symbols, and concretely, using ordered pair symbols. Composition tables for each SW-set are given in appendix 2; these are combined into one larger table for the entire SW-system in appendix 2a.

EXAMPLE 5-9: COMPOSITION TABLE FOR Q4

SW-SET Q_4

*	I	Q	-Q	QW	-QW
I	I	Q	-Q	QW	-QW
Q	Q	G	I	GW	W
-Q	-Q	I	-G	W	-GW
QW	QW	W	GW	I	G
-QW	-QW	-GW	W	-G	I

*	I	Q	-Q	QW	-QW
I	(C, +)	(G, +)	(F, +)	(G, -)	(F, -)
Q	(G, +)	(D,+)	(C, +)	(D, -)	(C, -)
-Q	(F, +)	(C, +)	(Bb, +)	(C, -)	(Bb, -)
QW	(G, -)	(C, -)	(D, -)	(C, +)	(D,+)
-QW	(F, -)	(Bb, -)	(C, -)	(Bb, +)	(C, +)

Composition of two chord transformations always results in a transformation belonging to the SW-system.[33] The system is operational—as appendix 2a confirms—but this is hardly surprising given Riemann's use of all six interval classes (the possibility of other *Schritte* is excluded, and a direct connection between all major and minor chords is ensured). What is less apparent is that the SW-system is a group, or more stringent operational system than the ones we have been discussing; a group operator must also be closed on the set, but operator

[33] Note however that SW-sets are not operational; the composition of two elements in Q_4, for example, sometimes gives results that are outside of the set (i.e. Q * Q = G).

and set must satisfy other conditions as well. We summarize these below in our general definition of a mathematical group.[34]

A group is a set G plus an operator * that satisfies the following requirements:

1. Identity. There is a unique element I in G such that $a * I = a$ for each a in G. I is the identity element of G.
2. Inverses. For each a in G there is an element $(-a)$ of G such that $a * (-a) = I$; $(-a)$ is the inverse of a with respect to I.
3. Associativity. For a, b, c in G, $a * (b * c) = (a * b) * c$; the operator * is associative on G.

Besides these attributes, many groups are also commutative, meaning that the order in which two of their elements are composed has no effect on the outcome: An operator is said to be commutative on G if, for every a and b in G, $a * b = b * a$. Commutativity is not a requirement of group structure, but is so common that commutative groups—also called "abelian" groups[35]—may seem to be the rule instead of a specialized case of group structure. This "rule" does not apply to the SW-system for, as we have seen, the order of composition significantly affects the outcome (i.e. QW * Q ≠ Q * QW). The SW-system is therefore noncommutative.

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[34] There are countless treatments of group theory in the pure and applied mathematical literature. We have used V. V. Nikulin and I. R. Shafarevich, *Geometries and Groups* (Berlin: Springer-Verlag, 1987); Michael Henle, *A Combinatorial Introduction to Topology* (New York: Dover, 1979), 143–52; and Joseph Landin, *An Introduction to Algebraic Structures* (New York: Dover, 1969), 52–117. For an elegant introduction to groups see Sir Arthur Stanley Eddington, "The Theory of Groups," in *The World of Mathematics*, ed. J. R. Newman, vol. 3 (New York: Simon and Schuster, 1956), 1033–56.

[35] After Niels Abel (1802–1829), one of the founders of group theory. See E. T. Bell, *Men of Mathematics* (New York: Simon and Schuster, 1965), 307–26.

Let us consider some simple examples of group structure before proceeding with the SW-system. We have observed that (P, +)—the set of positive integers under addition—is an operational system. A moment's reflection will show that the operator '+' is not only closed on P, but is also associative on P; thus, 1 + (2 + 3) = (1 + 2) + 3. (P, +) is not a group, however, because it does not satisfy the identity or inverse requirements (nos. 1 and 2 above). The integer '0' is needed for the identity element, and the negative integers are needed as inverses. Only by redefining P as the full set of integers Z does one derive an operational system (Z, +) that is also a group.[36] Notice how things change when one substitutes multiplication for addition in this group: (Z, x) is still operational, since 'x' is closed on Z; 'x' is also associative on Z, and there is a unique identity element in Z (1 rather than 0). So far, so good. (Z, x) is not a group, however, since most integers lack a multiplicative inverse in Z. The inverse of 5 is $^1/_5$, that of 4 is $^1/_4$, and so on—with the exception of 1 (which is its own inverse) and 0 (which has no inverse), the multiplicative inverses of integers are all non-integer rationals, and therefore excluded from Z.

The fact that (Z, +) is a group and (Z, x) is not highlights the importance of the relation between set and operator; this relation alone determines whether or not a system is a group. To dramatize this point, we consider a case where set and operator are both undefined. The set consists of three elements {◆ ❤ ♠}, which interface through an operator ❸ according to the conventions summarized in Example 5-10. We needn't ask what "diamond," "heart," and "spade" stand for in this example, or concern

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[36] The fact that (Z, +) is a commutative (abelian) group is incidental to its status as a group.

ourselves with the exact meaning of "eight-ball." Through analysis of the relationships alone, we can see that the system is associative (♦ ⑧ ♥) ⑧ ♠ = ♦ ⑧ (♥ ⑧ ♠), the identity is ♥, and the inverses of ♦ ♥ ♠ are ♠ ♥ ♦, respectively.[37] The system is therefore a group.

EXAMPLE 5-10: GROUP STRUCTURE OF AN UNDEFINED SYSTEM

⑧	♦	♥	♠
♦	♠	♦	♥
♥	♦	♥	♠
♠	♥	♠	♦

5.11 Groups and Symmetry

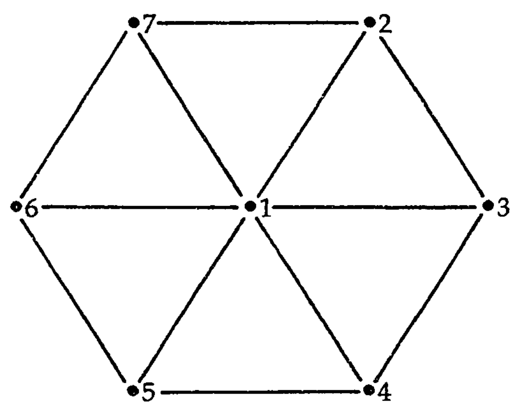
Wherever symmetry is found in art or nature there is a group related to it. Symmetry plays an important role in geometry and other branches of mathematics, and its study was an important force behind the development of group theory in the nineteenth century. The study of symmetry was not restricted to pure mathematics, but included important contributions from the physical sciences. There was keen interest in the symmetrical structure of crystals, for instance (the science of crystallography was born earlier in the century) which led to a formulation of several new groups.[38] The preoccupation with symmetry intensified in

[37] Ex. 5-10 is read just as the composition tables in Ex. 5-9 and app. 2: To find the result of composing "diamond" with "heart" (♦ ⑧ ♥), one finds ♦ in the column under ⑧ (the group operator) and reads across to the ♥ column. The point of intersection between row ♦ and column ♥ is ♦, therefore (♦ ⑧ ♥) = ♦. In like manner, one determines that the composition of "diamond" and "spade" is "heart" (♦ ⑧ ♠) = ♥. Therefore, (♦ ⑧ ♥) ⑧ ♠ = ♦ ⑧ ♠ = ♥. Similarly, ♦ ⑧ (♥ ⑧ ♠) = ♦ ⑧ ♠ = ♥, and (♦ ⑧ ♥) ⑧ ♠ = ♦ ⑧ (♥ ⑧ ♠).

[38] See Morris Kline, *Mathematics: The Loss of Certainty* (New York: Oxford Univ. Press, 1980), 294–95.

the latter half of the nineteenth century, when Hauptmann, Oettingen, and Riemann were all working out their music-theoretical ideas. We cannot begin to probe the relation between this scientific trend and contemporary trends in music theory, but clearly the influence on theorists was very great. Riemann declares in the first sentence of his *Modulationslehre* that "music is the art of symmetry in succession, just as architecture is the art of symmetry in juxtaposition."[39] The group structure of the SW-system is largely a consequence of the symmetries of the Table of Relations. We begin by considering some symmetries of the figure shown in Example 5-11.

EXAMPLE 5-11: HEXAGONAL SYMMETRY

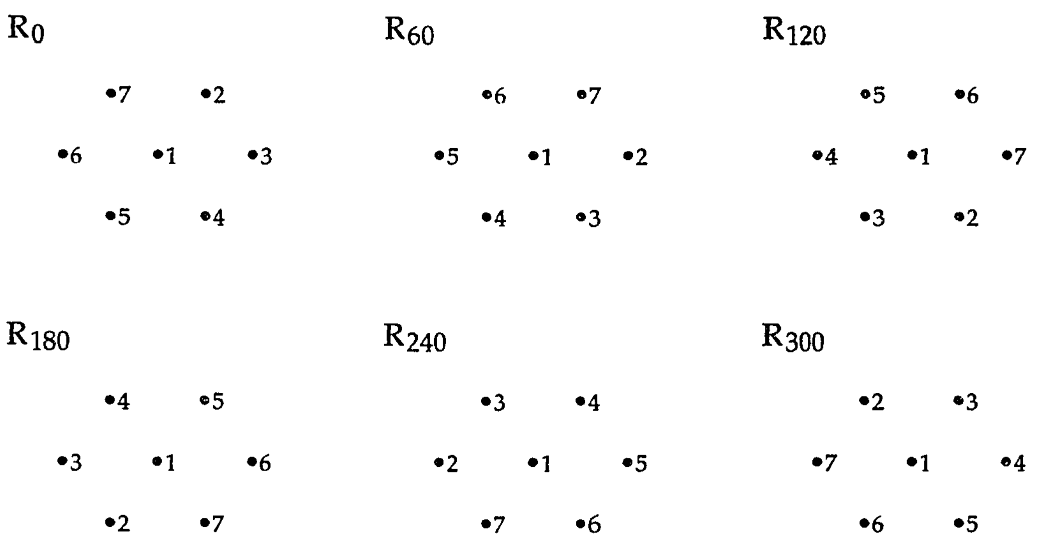


A symmetry of this figure is a transformation that preserves the distances between its points. We identify six of these transformations in Example 5-12—R_0, R_{60}, R_{120}, R_{180}, R_{240}, R_{300} (R_0 is an identity

[39] Riemann, *Systematische Modulationslehre*, 1. [Musik ist die Kunst der Symmetrie im Nacheinander, wie die Architektur die Kunst der Symmetrie im Nebeneinander (Miteinander) ist.]

transformation). Though the positions of points are different in each case, the relationships remain the same: Points 2, 3, and 4 are always across from points 5, 6, and 7. Transformations such as these, which neither alter nor otherwise distort a geometric figure, are called "rigid motions." The particular rigid motion that produces the symmetries in Example 5-12 is called "rotation," because the outer points are rotated clockwise—sixty degrees at a time—around the center. In general, rotation is a transformation that assigns an "image" point to every point in the coordinate plane, by rotating the plane around a fixed center. The hexagon is said to have rotational symmetry because it is an image of itself under the transformations R_0, R_{60}, R_{120}, R_{180}, R_{240}, and R_{300}.

EXAMPLE 5-12: ROTATIONAL SYMMETRY OF THE HEXAGON



Rotation is not the only rigid motion that produces symmetries. By exchanging points 7 and 2 with 5 and 4 in R_0, we produce a new symmetry. This exchange amounts to flipping the hexagon over its horizontal axis; by similarly flipping the other symmetries of Example 5-12, we can derive the symmetries shown in Example 5-13. The usual term for flipping motions of this sort is "reflection": F_0 and R_0, F_{60} and R_{60},...F_{300} and R_{300} are horizontal reflections of each other. We can grasp the idea more precisely by imagining line segments connecting points 7 and 5, 2 and 4, and 6, 1, 3 in R_0. Since the segment 6–1–3 bisects 7–5 and 2–4, and is perpendicular to them, it is called the "perpendicular bisector." It is also the line of symmetry, since the reflected points on 7–5 and 2–4 are equidistant to it. In general, a reflection assigns one point P to an image point P′ as follows: If P is on the line of symmetry, then P′ = P (P is its own image); if P is off the line of symmetry, then the line of symmetry is the perpendicular bisector of P–P′. The hexagon possesses reflectional symmetry because it is an image of itself under F_0, F_{60}, F_{120}, F_{180}, F_{240}, and F_{300}.

EXAMPLE 5-13: REFLECTIONAL SYMMETRY OF THE HEXAGON

F_0

•5 •4
•6 •1 •3
•7 •2

F_{60}

•4 •3
•5 •1 •2
•6 •7

F_{120}

•3 •2
•4 •1 •7
•5 •6

F_{180}

•2 •7
•3 •1 •6
•4 •5

F_{240}

•7 •6
•2 •1 •5
•3 •4

F_{300}

•6 •5
•7 •1 •4
•2 •3

The symmetries of Examples 5-12 and 5-13—the only ones that can result through rigid motion—comprise the "symmetry group" of the hexagon. We shall not bother with the composition table for this group. If we consider the rotations alone, however, we see that they form their own "rotation group." (The reflections do not form a self-contained group.)[40] Since the rotation group functions within the larger symmetry group, it is called a subgroup. We give its composition table in Example 5-14.

EXAMPLE 5-14: ROTATIONAL SUBGROUP OF THE HEXAGON

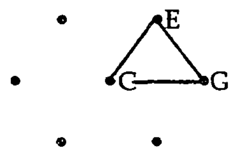
*	R0	R60	R120	R180	R240	R300
R0	R0	R60	R120	R180	R240	R300
R60	R60	R120	R180	R240	R300	R0
R120	R120	R180	R240	R300	R0	R60
R180	R180	R240	R300	R0	R60	R120
R240	R240	R300	R0	R60	R120	R180
R300	R300	R0	R60	R120	R180	R240

Let us now imagine the hexagon of Example 5-11 as a segment of the Table of Relations. By reinterpreting points 1, 2, and 3 as root, third, and fifth of a major triad, and viewing rotation as chord transformation, we can relate the rotational subgroup to different members of Riemann's SW-system. Example 5-15 charts the rotational transformations of a C major triad, and generalizes the results in a composition table.

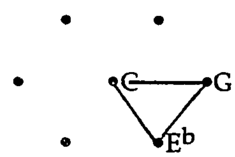
40 Unless they are construed as rotations, moving counterclockwise from F_0 to F_{300}. In this case they would form a group, whereas their "reflections" in Ex. 5-12 would not.

EXAMPLE 5-15: ROTATIONAL CHORD TRANSFORMATIONS

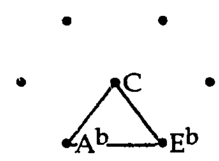
$R_0 = I$



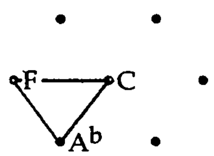
$R_{60} = -QW$



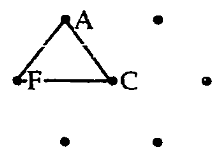
$R_{120} = -T$



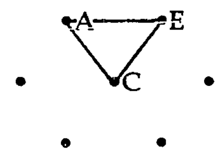
$R_{180} = W$



$R_{240} = -Q$



$R_{300} = TW$



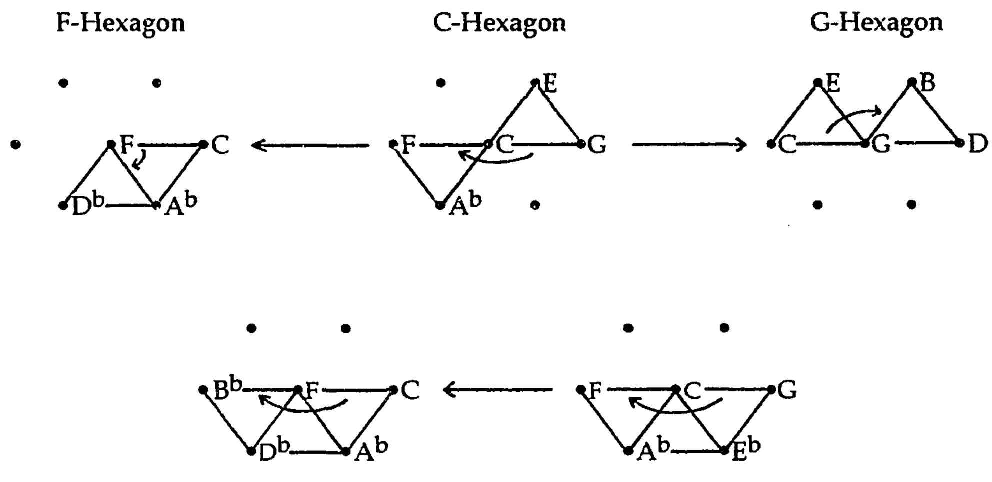
*	R0	R60	R120	R180	R240	R300
R_0	I	-QW	-T	W	-Q	TW
R_{60}	-QW	-T	W	-Q	TW	I
R_{120}	-T	W	-Q	TW	I	-QW
R_{180}	W	-Q	TW	I	-QW	-T
R_{240}	-Q	TW	I	-QW	-T	W
R_{300}	TW	I	-QW	-T	W	-Q

One problem with using hexagon rotations to derive chords in this manner is that too many chords are excluded from the process. We could remedy this by simply adding hexagons, so as to create a system of interlocking rotations. For example, if we added hexagons on F and G, we would have access to fourteen chords altogether. A move from C major to D^b major would be accomplished by rotating the C-hexagon 180° (to F minor), and then rotating the F-hexagon 60° (to D^b major). A move from C major to G major would involve rotation of the G-hexachord by 120°. We show these possibilities in Example 5-16a. Riemann did not use rotation—much less interlocking rotation—to derive chords, but the idea is consistent with his reliance on other forms of rigid motion. There are even cases where rotation seems a better way of navigating the Table; for example, it seems simpler to rotate through a series of descending thirds, as in Example 5-16b (C minor, A^b major, F minor, D^b major, Bb minor), than to apply alternating LW and TW relations: (G, -) –LW–> (A^b, +) –TW–> (C, -) –LW–> (D^b, +) –TW–> (F, -)

EXAMPLE 5-16: INTERLOCKING ROTATIONS

a)

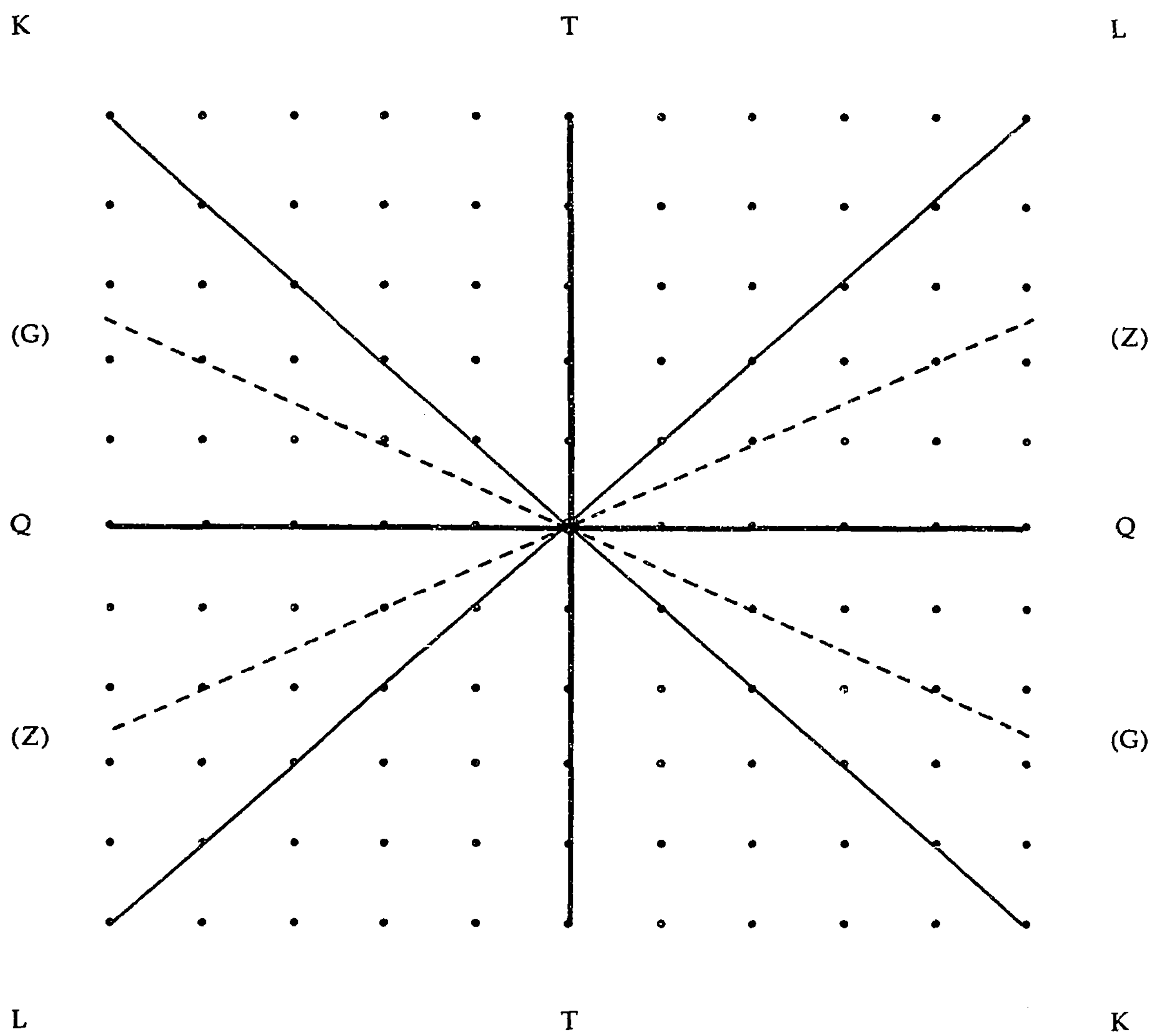
b)



We have spoken of "motion toward" and "motion around" in connection with harmonic *Schritte* and *Wechsel*. These motions, though less extravagant than the rotations in Example 5-16, are both rigid motions when viewed from the standpoint of the Table. *Wechsel* relations are reflections, or "motions around" a line of symmetry. *Schritte* are "motions toward" in a sense that we have not yet defined; they are uniform motions or "translations" along a straight line. In general, a translation is a rigid motion that "moves" every point in the plane the same distance and in the same direction. We have seen that the SW-system is closed, that it satisfies the identity and inverse requirements, and that it is noncommutative (though this is incidental to its group structure). What remains is to show that the system is associative. We shall do so now, with a modified version of the Table that allows us to plot chord reflections and translations abstractly (without reference to actual pitch) along six axes of transformation.

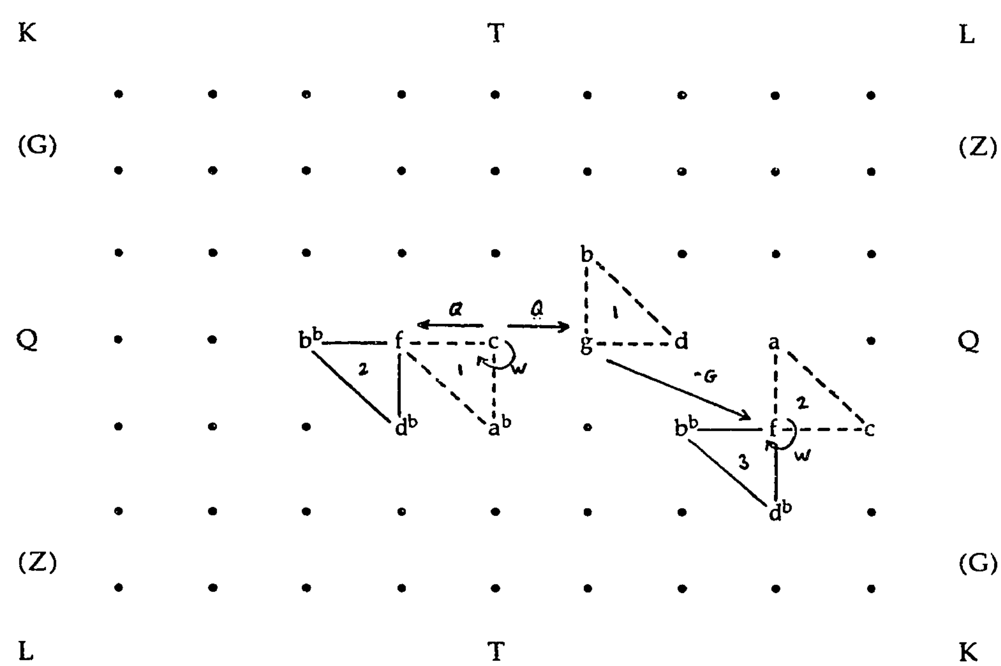
Our modified Table is presented in Example 5-17. This table never appeared in Riemann's work, but it reflects the main features of the SW-system. The horizontal and vertical axes are labeled Q (*Quint*) and T (*Terz*), respectively, and measure off fifth and third relations. Since these are the most important relations in Riemann's system, Q and T are the most prominent of the six axes. Two diagonal axes of less importance are labeled K (*Kleinterz*) and L (*Leitton*); these measure off minor third and minor second relations. Finally, two dotted axes labeled Z (*Tritonus*) and G (*Ganzton*) measure off tritone and wholetone relations. These axes are dotted because wholetone and tritone relations are very infrequent in Riemann's system. In general, any motion between major triads on T, K, L, G, and Z *above* the Q axis is a *schlichter Schritt*, and any motion *below* Q is a *gegen Schritt*. The opposite holds for minor triads. Motions right of the origin on Q are *schlichte Quintschritte* for major triads and *Gegenquintschritte* for minor triads; motions left of the origin are *Gegenquintschritte* for major triads and *schlichte Quintschritte* for minor triads.

EXAMPLE 5-17: AXES OF TRANSFORMATION



To demonstrate associativity, let us consider the transformation series Q * -QW * Q. If the SW-system is associative, then the manner in which we group these elements should not affect the outcome: Q * (-QW * Q) should come out the same as (Q * -QW) * Q. First we simplify. The composition table in appendix 2a states that -QW * Q = -GW, and that Q * -QW = W. By substituting -GW for (-QW * Q) and W for (Q * -QW), we may restate the relation as follows: Q * (-GW) = (W) * Q. The composition table states further that Q * (-GW) = -QW, and (W) * Q = -QW; therefore, Q * (-QW * Q) = (Q * -QW) * Q. We can visualize this relationship more clearly using the Table; a translation of Q, followed by a translation of -G, followed by a reflection shows the net result of Q *(-GW); a reflection around the origin, followed by a translation of Q shows the net result of (W) * Q. Both of these processes are sketched in Example 5-18.[41]

EXAMPLE 5-18: ASSOCIATIVITY ON THE TABLE



[41] Three versions of the KW relation are plotted in app. 2, and provide another example of associativity.

Notice that the algebraic equivalence of Q * (-GW) and (W) * Q in this example, is not borne out spatially; we are taken to two different locations on the Table. This inconsistency raises questions about enharmonic equivalence, and about the group structure of the SW-system. Clearly one could generate all twenty-four major and minor triads with fewer transformations than the number we have been considering; with just Q and W, for example, all twenty-four triads are accessible along the Table's Q-axis. If we limit ourselves to these transformations, and modularize the Table so that 6Q = -6Q ($F^{\#} = G^{b}$ where C is taken as the origin), combinations of Q and W that yield the same triad will point to the same location in the pitch space, and the inconsistency we have noted will disappear. Why does Riemann bother with other transformations if their effects can be mimicked by applications of Q and W in a modular space that assigns one location to each triad? This is a roundabout way of asking why he prefers the Table of Relations to the circle-of-fifths (or to any modular space with relatively few transformations). The answer is that Q and W, while sufficient to generate all twenty-four triads, do not permit important qualitative distinctions: 4Q and T yield enharmonically equivalent chords, but Riemann deliberately avoids characterizing these and other "equivalencies" in terms of Q (and W) alone. The SW-system is the fruit of a consistently inclusive attitude toward chord transformation (an attitude that Riemann does not reconcile with his stricter beliefs concerning the economy of perception). Instead of getting by with just two transformations, the system incorporates twenty-six.

We said earlier that the SW-system is group of transformations on the set SPACE; since the order of this group does not divide evenly into that of the set SPACE—26 does not divide evenly into 24—there must be redundancies in our formalization. The rogue elements have already been mentioned: -Z and -ZW generate the same triads as their *schlichter* counterparts Z and ZW. We might have excluded these "redundancies" from our account, but have deferred to Riemann's inclusion of them in *Handbuch.* Tritone-related transformations are judged in that late work to be distinct enough to warrant separate (albeit slight) treatment.

Unlike modular pitch spaces, the Table of Relations captures the qualitative difference of theoretically "equivalent" harmonic processes. The progressions indicated above by Q * (-GW) and (W) * Q sound very different, but their endpoints—(F, -) if the origin is set as C[42]—are enharmonically the same. Example 5-18 plots the musical difference, but we should stress that Riemann also recognized a kind of spatial equivalence between enharmonic chords and key areas. In his analysis of the Adagio from Beethoven's Piano Sonata in C minor op. 13, for example, he introduced the notion of direct leaps (*Überspringen*) between enharmonic pairs (keys in this case).[43] Such "plötzliche überspringen" lie outside the SW-system, and suggest that in enharmonic contexts there were ripples in Riemann's pitch space that permitted disjunct and potentially far-reaching motions. Some of the geometric possibilities for C are given in *Systematische Modulationslehre,* in the Table that we reproduce in Example 5-19. The related pitches, which Riemann frames, are widely separated on the Table; a move from 'c' to 'his^2', for example, is equivalent to two steps on the Z-axis of Example 5-17. Notice too that Riemann does not use *Striche* to differentiate pitches, a sure and

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[42] Q * (-GW) = (G, +) * (-GW) = (F, -); and (W) * Q = (C, -) * Q = (F, -).

[43] Riemann, *Ludwig van Beethovens sämtliche Klavier-Solosonaten,* vol. 2 (Berlin: Max Hesses, 1919), 25.

significant sign that the Table no longer represented tuning relations.[44] The superscripted and subscripted numbers merely "zeigen die Anzahl der Terzschritte von der Mittelreihe aus an."[45]

EXAMPLE 5-19: THE TABLE FROM *SYSTEMATISCHE MODULATIONSLEHRE*

his^{4}	fisis^{4}	cisis^{4}	gisis^{4}	disis^{4}	aisis^{4}	eisis^{4}	hisis^{4}	^{3}[illegible]f
gis^{3}	dis^{3}	ais^{3}	eis^{3}	his^{3}	fisis^{3}	cisis^{3}	gisis^{3}	disis^{3}
e^{2}	h^{2}	fis^{2}	cis^{2}	gis^{2}	dis^{2}	ais^{2}	eis^{2}	his^{2}
c^{1}	g^{1}	d^{1}	a^{1}	e^{1}	h^{1}	fis^{1}	cis^{1}	gis^{1}
as	es	b	f	c	g	d	a	e
$_{1}$fes	$_{1}$ces	$_{1}$ges	$_{1}$des	$_{1}$as	$_{1}$es	$_{1}$b	$_{1}$f	$_{1}$c
$_{2}$deses	$_{2}$asas	$_{2}$eses	$_{2}$heses	$_{2}$fes	$_{2}$ces	$_{2}$ges	$_{2}$des	$_{2}$as
$_{3}{}^{7}$h	$_{3}$feses	$_{3}$ceses	$_{3}$geses	$_{3}$deses	$_{3}$asas	$_{3}$eses	$_{3}$heses	$_{3}$fes

[44] The Table used in "Die Natur der Harmonik" still included *Striche* (see 3.15).
[45] Riemann, *Systematische Modulationslehre*, 172. This is the first work in which Riemann uses the Table to map key relations. He includes several "modulation tables" (pp. 172–73), whose *Klangschlüssel* represent the tonic triads of various keys. The most closely related keys encircle the home key on these tables. This principle of *Umschreibung* or key "circumscription" is important to Riemann. The actual tonic needn't be present in order to be felt, as long as the surrounding tonalities point to it. At the end of the *Modulationslehre* (pp. 204–07), Riemann explores this idea in connection with Beethoven's Piano Sonata in C major, op. 53, for which he provides several modulation tables.

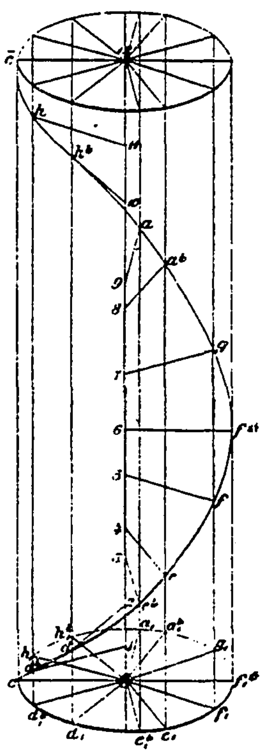
5.12 Conclusion

We conclude our study by considering briefly some psychological implications of Riemann's pitch space, and of musical symmetry in general. We have seen that the Table fulfilled a number of functions for Riemann, involving issues of intonation in his early work, and increasingly complex harmonic relations in later works. We have tried to convey a sense of correspondence between Riemann's notion of harmony at different stages of his career, and the Table of Relations. Even where the Table is not literally present in a treatise or pedagogical work, its structure looms over Riemann's conception of key and chord relations. The SW-system is so interwined with the geometry of the Table that it is difficult to imagine its evolution, and therefore its existence, independently of the Table. To the extent that theorists and historians have grappled with Riemann, without considering the Table and its traditions, they have failed to meet him on fundamental ground. Many apparent convolutions or oddities of thought—*Wechsel* relations and *Unterklänge* for example—become transparent when seen in the light of the Table. We do not claim that the Table justifies Riemann's theories, but that it is the most important means toward understanding them.

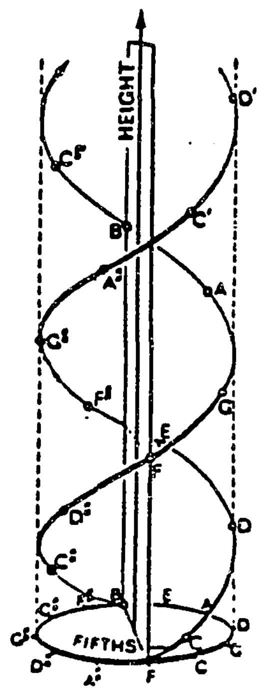
We have reduced the Table to a geometrical space consisting of just two modes of transformation—translation and reflection—in order to get at the heart of Riemann's conception. This conception is strikingly visual (as was Oettingen's and Hauptmann's); triangular figures are mentally translated or reflected along geometrical axes in the plane. The idea of conceiving sounds as objects that in some literal sense may be "turned over" in one's mind, is not unique to Riemann's work. Numerous spatial-geometrical models of pitch emerged in the late nineteenth century, in diverse studies of acoustics, tone psychology, and music theory.[46] We cite one of these below, in Example 5-20a, which appeared in a mid-century work by the prolific philosopher-scientist M. W. Drobisch.[47]

EXAMPLE 5-20: DROBISCH'S SPIRAL MODEL AND SHEPARD'S "DOUBLE HELIX"

a) Drobisch (1852)



b) Shepard (1982)



Drobisch's pitch space is clearly different from Riemann's; translation and reflection are the operative moves in Riemann's model, whereas rotation is the natural choice in Drobisch's. What these models have in

[46] See Marvin, "*Tonpsychologie* and *Musikpsychologie*."
[47] Drobisch, *Über musikalische Tonbestimmung und Temperatur*,121.

common is the assumption of a geometric substrate against which auditory perceptions are claimed to be organized and interpreted. Much recent work in cognitive psychology has focused on the geometric modeling of auditory and visual phenomena. Shepard (1982) has proposed a well-known double helical model of pitch, reproduced in Example 5-20b, which is derived from Drobisch's spiral model.[48] Krumhansl and Shepard (1979) have used a four-dimensional torus to map pitch relations.[49] The thrust behind such work is the belief that "similar abstract geometrical structures" are a consequence of the "rotational transformations and symmetries that we presume to underlie the perceived relations among the objects in both domains alike."[50] Not everyone has presumed so much about underlying symmetries and transformations in the domain of music perception. Riemann's great contemporary Ernst Mach wrote that symmetry "such as is found in the province of sight, cannot be imagined in music, since sensations of tone do not constitute a symmetrical system."[51] Perhaps, in time, the research of cognitive psychologists will allow us to inquire more reliably into the transformations of Riemann's system, and the psychological validity of the Table of Relations. But whether or not we uphold Mach's judgment in the end, the Table will remain a fascinating gateway to a rich and still vital music-theoretical tradition.

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[48] Roger N. Shepard, "Structural Representations of Musical Pitch." In *The Psychology of Music*, ed. Diana Deutsch (New York: Academic Press, 1982).

[49] Roger N. Shepard and Carol L. Krumhansl, "Quantification of the Hierarchy of Tonal Functions Within a Diatonic Context" *Journal of Experimental Psychology: Human Perception and Performance* 5 (1979): 578–94.

[50] Roger N. Shepard and Lynn A. Cooper, *Mental Images and Their Transformations* (Cambridge, Mass.: MIT Press, 1986), 321.

[51] Ernst Mach, *Die Analyse der Empfindungen und das Verhältnis des Physischen zum Psychischen*, 6th ed. (Jena: Gustav Fischer Verlag, 1911), 222 n. 2. [An eine vollwerthige Symmetrie wie im Gebiete des Gesichtsinnes darf natürlich im Gebiete der Musik, da die Tonempfindungen selbst kein symmetrisches System bilden, nicht gedacht werden.]

BIBLIOGRAPHY

Aldwell, Edward and Carl Schachter. *Harmony and Voice Leading.* 2d ed. New York: HBJ, 1989.

Arens, Katherine. *Structures of Knowing: Psychologies of the Nineteenth Century.* Boston Studies in the Philosophy of Science, vol. 113. Dordrecht: Kluwer Academic Publishers, 1989.

Auerbach, Felix. *Die Grundlagen der Musik.* Leipzig, 1911.

Barbour, J. M. "Musical Logarithms." *Scripta Mathematica* 7 (1940): 21–31.

Barker, Andrew, ed. *Greek Musical Writings: II. Harmony and Acoustic Theory.* Cambridge: Cambridge Univ. Press, 1989.

Bell, E. T. *Men of Mathematics.* New York: Simon and Schuster, 1965.

Bellerman, Heinrich. *Der Contrapunkt; oder, Anleitung zur Stimmführung in der musikalischen Composition.* Berlin: Julius Springer, 1862.

Bernstein, David W. "Symmetry and Symmetrical Inversion in Turn-of-the-Century Theory and Practice." In *Music Theory and the Exploration of the Past,* edited by Christopher Hatch and David W. Bernstein. Chicago and London: Univ. of Chicago Press, 1993.

———. "Schoenberg and the Austro-German Theoretic Legacy." Paper presented at Schoenberg Conference, Columbia University, New York, October 1992.

———. "Schoenberg Contra Riemann: *Stufen,* Regions, *Verwandschaft,* and the Theory of Tonal Function."*Theoria* 6 (1992): 23–53.

Boetticher, Wolfgang. "Eduard Krüger als Professor der Musikgeschichte an der Georgia Augusta." In *Musikwissenschaft und Musikpflege an der Georg-August-Universität Göttingen: Beiträge in ihrer Geschichte,* edited by Martin Staehelin. Göttingen: Vandenhoeck & Ruprecht, 1987.

Boltzmann, Ludwig. *Populäre Schriften.* 3d ed. Leipzig: J. A. Barth, 1925.

Botstein, Leon. "Time and Memory: Concert Life, Science, and Music in Brahms's Vienna." In *Brahms and His World,* edited by Walter Frisch. Princeton: Princeton Univ. Press, 1990.

Burnham, Scott. "Method and Motivation in Hugo Riemann's History of Harmonic Theory."*Music Theory Spectrum* 14/1 (1992): 1–14.

Busch, Hermann R. *Leonhard Eulers Beitrag zur Musiktheorie.* Regensburg: Gustav Bosse Verlag, 1970.

Caplin, William. "Tonal Function and Metrical Accent." *Music Theory Spectrum* 5 (1983): 1–14.

Chomsky, Noam. *Syntactic Structures.* The Hague: Mouton, 1957.

Christensen, Thomas. *Rameau and Musical Thought in the Enlightenment.* Cambridge: Cambridge Univ. Press, 1993.

Dahlhaus, Carl. *Untersuchungen über die Entstehung der harmonischen Tonalität.* Kassel: Bärenreiter, 1967. Translated by Robert O. Gjerdingen as *Studies on the Origin of Harmonic Tonality.* Princeton: Princeton Univ. Press, 1990.

Descartes, René. *Compendium of Music.* Translated by Walter Robert with an introduction and notes by Charles Kent. Musicological Studies and Documents, no. 8. Rome: American Institute of Musicology, 1961.

Devore, Richard. "Nineteenth-Century Harmonic Dualism in the United States." *Theoria* 2 (1987): 85–100.

Drobisch, Moritz Wilhelm. *Über musikalische Tonbestimmung und Temperatur.* Leipzig: Weidmann, 1852.

———. *Neue Darstellung der Logik nach ihrer einfachsten Verhältnisse, mit Rücksicht auf Mathematik und Naturwissenschaft,* 2d ed. Hamburg und Leipzig: Leopold Voss, 1851; 3/1863; 4/1875; 5/1887.

Dufner, Max, and Valentine C. Hubbs, eds. *German Essays II.* MacMillan: New York, 1964.

Eddington, Sir Arthur Stanley. "The Theory of Groups." In *The World of Mathematics,* edited by James R. Newman, vol. 3. New York: Simon and Schuster, 1956.

Ellis, Alexander J., trans. and ed. *On the Sensations of Tone as a Physiological Basis for the Theory of Music by Hermann Helmholtz*. London: Longmans, 1875. Reprint. New York: Dover, 1954.

Encylopaedia Britannica, 15th ed. S.v. "Helmholtz."

Euler, Leonhard. *Leonhardi Euleri Opera omnia sub auspiciis Societatis scientiarum naturalium Helveticae.* Ser. 3, no. 1, edited by Eduard Bernoulli, Rudolph Bernoulli, Ferdinand Rudio, and Andreas Speiser. Leipzig and Berlin: B. G. Teubner, 1926; Ser. 3, nos. 11–12, edited by Andreas Speiser. Zurich: Orell Füssli, 1960.

———. "Musices theoreticae systema." Unpublished notebook, ca. 1726.

———. *Dissertatio physica de sono.* Basel: typis E. & J. R. Thurnisiorum, 1727.

———. *Tentamen novae theoriae musicae ex certissimis harmoniae principiis dilucide expositae.* Petropoli: Ex typographia Academiae scientiarum, 1739. Facsimile edition. New York: Broude, 1968.

———. "Conjecture sur la raison de quelques dissonances généralement reçues dans la musique." *Mémoires de l'academie des sciences de Berlin* 20 (1764): 165–73.

———. "Du véritable caractère de la musique moderne." *Mémoires de l'academie des sciences de Berlin* 20 (1764): 174–99.

———. *Lettres à une princesse d'Allemagne sur divers sujets de physique et de philosophie* [1760–62]. St. Petersburg: Imprimerie de l'Académie impériale des sciences, 1768–72.

———. "De harmoniae veris principiis per speculum musicum repraesentatis," *Novi commentarii academiae scientarium Petropolitanae* 18 (1773): 330–53.

Fancher, Raymond E. *Pioneers of Psychology.* New York: Norton, 1979.

Farnsworth, Paul R. *The Social Psychology of Music.* Ames, Iowa: Iowa State Univ. Press, 1969.

Farrell, R. B. *A Dictionary of German Synonyms.* London: Cambridge Univ. Press, 1968.

Fechner, Gustav T. *Vorschule der Aesthetik.* Leipzig: Breitkopf und Härtel, 1876. Reprint. Hildesheim: Georg Olms, 1978.

Fokker, Adrian D., and Jan von Dijk. "Expériences musicales avec les genres musicaux de Leonhard Euler contenant la septième harmonique." In *Internationale Gesellschaft für Musikwissenschaft, Kongressbericht.* Basel: Barenreiter, 1949.

Francès, Robert. *La perception de la musique.* Paris: Librairie philosophique J. Vrin, 1958. Translated by W. Jay Dowling as *The Perception of Music.* Hillsdale, N.J.: Lawrence Erlbaum Associates, Publishers, 1988.

Fuß, Nicolaus. "Lobrede auf Herrn Euler...an 23 Oktober vorgelesen." In *Leonhardi Euleri Opera omnia sub auspiciis Societatis scientiarum naturalium Helveticae.* Ser. 1, no. 1, edited by Eduard Bernoulli, Rudolph Bernoulli, Ferdinand Rudio, and Andreas Speiser. Leipzig and Berlin: B. G. Teubner, 1911.

Gay, Peter. *Freud, Jews and Other Germans. Masters and Victims in Modernist Culture.* New York: Oxford Univ. Press, 1978.

Goethe, Johann Wolfgang von. *Gedenkausgabe der Werke, Briefe und Gespräche.* 2d ed., edited by Ernst Beutler. Vol. 17, *Naturwissenschaftliche Schriften.* Zürich: Artemis Verlags, 1966.

———. *Sämtliche Werke: Briefe, Tagebücher und Gespräche,* edited by Manfred Wenzel. Vol. 23/1, *Zur Farbenlehre. Das Gesamte Hauptwerk von 1810.* Frankfurt am Main: Deutscher Klassiker Verlag, 1991.

———. *Conversations with Eckermann.* Translated by John Oxenford. San Francisco: North Point Press, 1984.

Goldschmidt, Ludwig. *Kant und Helmholtz: Populärwissenschaftliche Studien.* Hamburg und Leipzig: Leopold Voss, 1898.

Gurlitt, Wilibald. "Hugo Riemann (1849–1919)." *Akademie der Wissenschaften und der Literatur in Mainz: Abhandlungen der Geistes und Sozialwissenschaftlichen Klasse* 25 (1950): 1863–1905.

Hamm, Josef. *Das philosophische Weltbild von Helmholtz.* Bonn: Beyer, 1937.

Hanslick, Eduard. *Vom Musikalisch-Schönen: ein Beitrag zur Revision der Ästhetik der Tonkunst.* Leipzig, 1854. Reprint. Darmstadt: Wissenschaftliche Buchgesellschaft, 1981.

Harrison, Daniel. *Harmonic Function in Chromatic Music: A Renewed Dualist Theory and an Account of Its Precedents.* Chicago and London: Univ. of Chicago Press, 1994.

Hatfield, Gary C. *The Natural and the Normative: Theories of Spatial Perception from Kant to Helmholtz.* Cambridge, MA.: MIT Press, 1990.

Hauptmann, Moritz. *Die Natur der Harmonik und der Metrik: Zur Theorie der Musik.* Leipzig: Breitkopf und Härtel, 1853; 2/1873.

Heath, Sir Thomas L. *A Manual of Greek Mathematics.* New York: Dover, 1963.

Helmholtz, Hermann von. *Die Lehre von den Tonempfindungen als physiologische Grundlage für die Theorie der Musik.* Braunschweig: Fr. Vieweg & Sohn, 1863; 2/1865; 3/1870; 4/1877; 5/1896; 6/1913. Reprint. of 6th ed. Hildesheim: Georg Olms, 1968.

———. *Handbuch der physiologischen Optik.* Hamburg und Leipzig: Leopold Voss, 1867 (parts appeared between 1856 and 1867); 2/1896; 3/1909–11. Translation of 3d ed. (3 vols.) as *Helmholtz's Treatise on Physiological Optics.* New York: Optical Society of America, 1924–25. Reprint (3 vols. as 2), edited by J. P. C. Southall. New York: Dover, 1962.

———. *Die Thatsachen in der Wahrnehmung: Rede gehalten zur Stiftungsfeier der Friedrich-Wilhelms-Universität zu Berlin am 3. August 1878.* Berlin: August Hirschwald, 1879.

———. *Über die Erhaltung der Kraft.* Berlin: G. Reimer, 1847. Reprint. *Wissenschaftliche Abhandlungen von Hermann Helmholtz,* vol. 1. Leipzig: J. A. Barth, 1882.

Henle, Michael. *A Combinatorial Introduction to Topology.* New York: Dover, 1979.

Hyer, Brian. "Tonal Intuitions in "*Tristan und Isolde.*" Ph.D. diss., Yale University, 1989.

Imig, Renate. *Systeme der Funktionsbezeichnung in den Harmonielehren seit Hugo Riemann.* Orpheus-Schriftenreihe zu Grundfragen der Musik, vol. 9. Düsseldorf: Gesellschaft zur Förderung der systematischen Musikwissenschaft, 1970.

Janik, Allan and Stephen Toulmin. "Language, Ethics and Representation." Chap. 5 in *Wittgenstein's Vienna.* New York: Simon and Schuster, 1973.

Johnston, William M. *The Austrian Mind: An Intellectual and Social History, 1848–1938.* Berkeley: Univ. of California Press, 1972.

Jonas, Oswald. *An Introduction to the Theory of Heinrich Schenker: The Nature of the Musical Work of Art.* Longman: New York, 1982.

Kant, Immanuel. *Critique of Pure Reason* Translated by Norman Kemp Smith. New York: St. Martin's Press, 1965.

Kaufmann, Walter. *Goethe, Kant, and Hegel.* Vol. 1 of *Discovering the Mind.* New Brunswick, N.J.: Transaction Publishers, 1991.

Kline, Morris. *Mathematics: The Loss of Certainty.* New York: Oxford Univ. Press, 1980.

Klumpenhouwer, Henry. "Some Remarks on the Use of Riemann Transformations." *Music Theory Online* 0/9 (1994).

Komar, Arthur. *Linear-Derived Harmony.* Cincinatti: Overbird Press, 1992.

Korsyn, Kevin. "Schenker's Organicism Reexamined." Paper read at the Second International Schenker Symposium, Mannes College of Music, New York, March 1992.

Krüger, Felix. "Die Theorie der Konsonanz." *Psychologische Studien* 4 (1909).

Kuhn, Thomas. *The Structure of Scientific Revolutions.* 2d ed. Chicago: Univ. of Chicago Press, 1970.

Landin, Joseph. *An Introduction to Algebraic Structures.* New York: Dover, 1969.

Leibniz, Gottfried Wilhelm. *Epistolae ad diversos.* Edited by Christian Kortholt. Leipzig: Bern. Christoph Brettkopf, 1734–42.

Lerdahl, Fred, and Ray Jackendoff. *A Generative Theory of Tonal Music.* Cambridge, MA.: MIT Press, 1983.

Lerdahl, Fred. "Tonal Pitch Space." *Music Perception* 5/3 (1988): 315–49.

Leshan, Lawrence, and Henry Margenau. *Einstein's Space and Van Gogh's Sky.* New York: Macmillan, 1982.

Lewin, David. "A Formal Theory of Generalized Tonal Functions." *Journal of Music Theory* 26/1 (1982): 23–60.

———. "Amfortas's Prayer to Titurel and the Role of D in *Parsifal*: The Tonal Spaces of the Drama and the Enharmonic C♭/B." *19th-Century Music* 7/3 (1984): 336–49.

———. *Generalized Musical Intervals and Transformations.* New Haven: Yale Univ. Press, 1987.

Lipps, Theodor. *Psychologische Studien.* Leipzig, 1905.

Lotze, Hermann. *Geschichte der Ästhetik in Deutschland.* Geschichte der Wissenschaften in Deutschland: Neuere Zeit, no. 7. Munich: J. G. Cotta, 1868.

Lorenz, Alfred. *Der Musikalische Aufbau von Richard Wagners "Tristan und Isolde."* Vol. 2 of *Das Geheimis der Form bei Richard Wagner.* Berlin: Max Hesses, 1926.

Lugo, Carlo del. *Goethe ed Helmholtz.* Torino, 1903.

Mach, Ernst. *Einleitung in die Helmholtz'sche Musiktheorie: Populär für Musiker dargestellt.* Graz: Leuschner und Lubensky, 1866.

———. *Die Analyse der Empfindungen und das Verhältnis des Physischen zum Psychischen.* 6th ed. Jena: Gustav Fischer, 1911. Translated by C. M. Williams as *The Analysis of Sensations and the Relation of the Physical to the Psychical.* New York: Dover, 1959.

Marvin, Elizabeth West. "*Tonpsychologie* and *Musikpsychologie.* Historical Perspectives on the Study of Music Perception." *Theoria* 2 (1987): 59–84.

Mattheson, Johann. "Die neue Zahl-Theorie." In *Plus ultra, ein Stückwerk von neuer und mancherley Art.* 4 vols. Hamburg: Johann Adolph Martini, 1754–56.

McCune, Mark. "Hugo Riemann's 'Ueber Tonalität': A Translation." *Theoria* 1 (1985): 132–50.

Meinong, Alexius, and St. Witasek. "Zur experimentellen Bestimmung der Tonverschmelzungsgrade." *Zeitschrift für Psychologie und Physiologie der Sinnesorgane* 15 (1897).

Meyer, Max. "Zur Theorie der Differenztöne und der Gehörsempfindungen überhaupt." In *Beiträge zur Akustik und Musikwissenschaft*, vol. 2. Leipzig: J. A. Barth, 1898.

Mickelsen, William C. *Hugo Riemann's Theory of Harmony and History of Music Theory, Book 3.* Lincoln: Univ. of Nebraska Press, 1977.

Mizler, Lorenz. *Neue eröffnete musikalische Bibliothek; oder, Grundliche Nachricht nebst unpartheyischem Urtheil von musikalischen Schriften und Büchern.* Leipzig: im Verlag des Verfassers und bey Brauns Erben in Commission, 1736–54. Facsimile edition. 3 vols. Hilversum: Frits Knuf, 1966.

Müller-Freienfels, Richard, ed. *Eislers Handwörterbuch der Philosophie.* 2d ed. Berlin: E. S. Mittler und Sohn, 1922. S.v. "Angeboren," "Nativismus," "Vorstellung," and "Darstellung."

Münnich, Richard. "Vom Entwicklung der Riemannschen Harmonielehre und ihrem Verhältnis zu Oettingen und Stumpf." In *Riemann-Festschrift: Gesammelte Studien.* Leipzig: Max Hesse, 1909. Reprint. Tützing: Schneider, 1965.

Nikulin, V. V. and I. R. Shafarevich. *Geometries and Groups.* Berlin: Springer, 1987.

Oettingen, Arthur von. *Harmonieystem in dualer Entwickelung: Studien zur Theorie der Musik.* Dorpat und Leipzig: W. Gläser, 1866.

———. "Das duale System der Harmonie." *Ostwalds Annalen der Naturphilosophie Jahrgänge* 1 (1902): 62–75; 2 (1902): 375–403; 3 (1904): 241–69; 4 (1905): 116–36, 301–38; 5 (1906): 449–503.

———. *Das duale Harmoniesystem.* Leipzig: C. F. W. Siegel's Musikalienhandlung (R. Linnemann), 1913.

———. "Die Grundlage der Musikwissenschaft und das duale Reininstrument." *Abhandlungen der mathematisch-physischen Klasse der Sächsischen Gesellschaft der Wissenschaften* 34/2 (1916): 155–361.

Peterson, Ivars. *The Mathematical Tourist: Snapshots of Modern Mathematics.* New York: W. H. Freeman, 1988.

Preyer, W. T. *Über die Grenzen der Tonwahrnehmung.* Jena: H. Dufft, 1876.

Quine, W. V. *Methods of Logic.* 4th ed. Cambridge, Mass.: Harvard Univ. Press, 1982.

Rasch, Rudolph. "Theory of Helmholtz-Beat Frequencies." *Music Perception* 1/3 (1984): 308–22

Riehl, Alois. *Führende Denker und Forscher.* 2d ed. Leipzig: Quelle und Meyer, 1922.

Riemann, Hugo [Hugibert Ries, pseud.]. "Musikalische Logik: Ein Beitrag zur Theorie der Musik." *Neue Zeitschrift für Musik* 68/28–29, 36–38 (1872): 279–82, 287–88, 353–55, 363–64, 373–74.

———. [Hugibert Ries, pseud.]. "Ueber Tonalität." *Neue Zeitschrift für Musik* 68/45–46 (1872): 443–44, 453–54.

———. "Ueber das musikalische Hören." Ph.D. diss., University of Göttingen, 1873. Printed twice under this title. Göttingen: Vandenhoeck & Ruprecht, 1874; Leipzig: F. Andrä, 1874. Retitled *Musikalische Logik: Hauptzüge der physiologischen und psychologischen Begründung unseres Musiksystems.* Leipzig: C. F. Kahnt, 1874.

———. *Musikalische Syntaxis: Grundriß einer harmonischen Satzbildungslehre.* Leipzig: Breitkopf und Härtel, 1877.

———. *Skizze einer neuen Methode der Harmonielehre.* Leipzig: Breitkopf und Härtel, 1880; 2d ed. enlarged as *Handbuch der Harmonielehre.* Leipzig: Breitkopf und Härtel, 1887; 3/1898; 4/1906; 5/1912; 6/1918.

———. *Musik-Lexicon: Theorie und Geschichte der Musik, die Tonkünstler alter und neuer Zeit mit Angabe ihrer Werke, nebst einer vollständigen Instrumentenkunde.* Leipzig: Bibliographisches Institut, 1882.

———. "Die Natur der Harmonik." *Sammlung musikalischer Vorträge: Ein Almanach für die musikalische Welt* 4/40 (1882): 157–90. Translated by John Fillmore Comfort as "The Nature of Harmony." In *New Lessons in Harmony*. Philadelphia: Presser, 1886.

———. *Systematische Modulationslehre als Grundlage der musikalischen Formenlehre*. Hamburg: J. F. Richter, 1887.

———. *Katechismus der Harmonielehre. (Praktische Anleitung zum mehrstimmigen Tonsatz)*. Leipzig: Max Hesses, 1890.

———. *Vereinfachte Harmonielehre; oder Die Lehre von den tonalen Funktionen der Akkorde*. London: Augener; New York: Schirmer, 1893. Translated by Henry Bewerunge as *Harmony Simplified; or, The Theory of the Tonal Functions of Chords*. London: Augener, 1896.

———. "Das chromatische Tonsystem." In *Präludien und Studien: Gesammelte Aufsätze zur Ästhetik, Theorie und Geschichte der Musik*. Vol 1. Frankfurt am Main: Bechhold, 1895.

———. *Die elemente der musikalischen Ästhetik*. Berlin und Stuttgart: W. Spemann, 1900.

———. "Zur Reform der Harmonie-Lehrmethode." In *Präludien und Studien: Gesammelte Aufsätze zur Ästhetik, Theorie und Geschichte der Musik*, vol. 3. Leipzig: Hermann Seeman, 1901.

———. *Große Kompositionslehre*. Vol. 1, *Der homophone Satz*. Berlin und Stuttgart: W. Spemann, 1902.

———. *Elementar-Schulbuch der Harmonielehre*. Leipzig: Max Hesse, 1906.

———. "Ideen zu einer 'Lehre von den Tonvorstellung.'" *Jahrbuch der Musikbibliothek Peters* 21–22 (1914/15): 1–26. Reprint. Wiesbaden: Lessing, 1965.

———. *Ludwig van Beethovens sämtliche Klavier-Solosonaten: Ästhetische und formal-technische Analyse mit historischen Notizen*. 3 vols. Berlin: Max Hesses, 1917–19.

Rothstein, William. *Phrase Rhythm in Tonal Music*. New York: Schirmer, 1989.

Rummenhöller, Peter. "Moritz Hauptmann, Der Begründer einer transzendental-dialektischen Musiktheorie." In *Beiträge zur Musiktheorie des 19. Jahrhunderts,* edited by Martin Vogel. Regensburg: Gustav Bosse, 1966.

Runes, Dagobert D., ed. *Dictionary of Philosophy.* New York: Philosophical Library, Inc., 1983. S.v. "nativism."

[S., J.?] "Ein neues Harmoniesystem." *Neue Zeitschrift für Musik* 65/42 (1869): 349–52. [Review of Oettingen's *Harmoniesystem in dualer Entwickelung.*]

Saslaw, Janna K. "Gottfried Weber and Multiple Meaning." *Theoria* 5 (1990–91): 74–103.

Schäfke, Rudolf. *Geschichte der Musikästhetik in Umrissen.* Berlin und Schöneberg: Max Hesse, 1934.

Schoenberg, Arnold. *Fundamentals of Musical Composition.* London: Faber and Faber, 1983.

Schorske, Carl. *Fin-de-Siècle Vienna: Politics and Culture.* New York: Vintage, 1981.

Schwertschlager, Joseph. *Kant und Helmholtz erkenntniss–theoretisch verglichen.* Freiburg im Breisgau: Herder'sche Verlagshandlung, 1883.

Seidel, Elmar. "Die Harmonielehre Hugo Riemanns." In *Beiträge zur Musiktheorie des 19. Jahrhunderts,* edited by Martin Vogel. Regensburg: Gustav Bosse, 1966.

Serre, Jean-Adam. *Essais sur les principes de l'harmonie.* Paris: Prault fils, 1753. Facsimile edition. New York: Broude, 1967.

Shepard, Roger N. "Structural Representations of Musical Pitch." In *The Psychology of Music,* edited by Diana Deutsch. New York: Academic Press, 1982.

Shepard, Roger N., and Carol L. Krumhansl. "Quantification of the Hierarchy of Tonal Functions Within a Diatonic Context." *Journal of Experimental Psychology: Human Perception and Performance* 5 (1979): 578–94.

Shepard, Roger N., and Lynn A. Cooper. *Mental Images and Their Transformations.* Cambridge, Mass.: MIT Press, 1986.

Shirlaw, Matthew. *The Theory of Harmony.* London: Novello, 1917. Reprint. Dekalb, Ill.: B. Coar, 1955.

Simmel, Georg. *Kant und Goethe.* Berlin: Bard, Marquardt und Co., 1906.

Smith, Charles Samuel. "Leonhard Euler's *Tentamen novae theoriae musicae*: A Translation and Commentary." Ph.D. diss., Indiana University, 1960.

Smith, Peter H. "Brahms and Schenker: A Mutual Response to Sonata Form." *Music Theory Spectrum* 16/1 (1994).

Stumpf, Carl. *Tonpsychologie.* 2 vols. Leipzig: S. Hirzel, 1883–90.

———. "Konsonanz und Dissonanz." In *Beiträge zur Akustik und Musikwissenschaft,* vol. 1. Leipzig: J. A. Barth, 1898.

Tarski, Alfred. *Introduction to Logic and to the Methodology of Deductive Sciences.* 2d ed. New York: Oxford Univ. Press, 1946.

Terhardt, Ernst. "The Concept of Musical Consonance: A Link Between Music and Psychoacoustics." *Music Perception* 1/3 (Spring 1984): 276–95.

Vogel, Martin. "Die Zahl Sieben in der spekulativen Musiktheorie." Ph.D. diss., University of Bonn, 1954.

———. "Die Musikschriften Leonhard Eulers." In *Leonhardi Euleri Opera omnia sub auspiciis Societatis scientiarum naturalium Helveticae.* Ser. 3, nos. 11–12, edited by Andreas Speiser. Zurich: Orell Füssli, 1960.

———. "Arthur v. Oettingen und der harmonische Dualismus." In *Beiträge zur Musiktheorie des 19. Jahrhunderts,* edited by Martin Vogel. Regensburg: Gustav Bosse, 1966.

———. *Anleitung zur harmonischen Analyse und zu reinen Intonation.* Bonn: Verlag für systematische Musikwissenschaft, 1984.

Vogler, Abbé George Joseph. *Handbuch zur Harmonielehre und für den Generalbaß, nach den Grundsätzen der Mannheimer Tonschule.* Prague: Karl Barth, 1802.

Warrack, Guy. "Music and Mathematics." *Music and Letters* 24 (1945).

Warren, Richard M. "Helmholtz and His Continuing Influence." *Music Perception* 1/3 (1984): 253–75

Warren, Richard M., and Roslyn P. Warren. *Helmholtz on Perception: Its Physiology and Development.* New York: Wiley, 1968.

Wason, Robert W. *Viennese Harmonic Theory from Albrechtsberger to Schenker and Schoenberg.* UMI Research Press: Ann Arbor, 1985.

Wason, Robert, and Elizabeth West Marvin, "Riemann's 'Ideen zu einer Lehre von den Tonvorstellungen': An Annotated Translation." *Journal of Music Theory* 36/1 (1992): 69–117.

Weber, Gottfried. *Versuch einer geordneten Theorie der Tonsetzkunst zum Selbstunterricht mit Anmerkungen für Gelehrtere.* 3 vols. Mainz: B. Schott, 1817–21; 3d rev. ed./1830–32 (4 vols.).

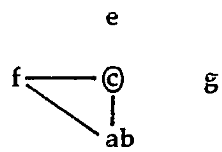
White, Harvey E., and Donald H. White. *Physics and Music: The Science of Musical Sound.* Philadelphia: Saunders College, 1980.

Wundt, Wilhelm. *Grundriß der Psychologie.* Leipzig: Alfred Kröner, 1908.

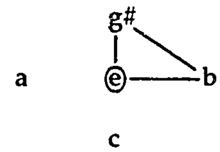
APPENDIX 1: THE SYSTEM OF *SCHRITTE* AND *WECHSEL*

I: Seitenwechsel

1) **Seitenwechsel = W**
(C, +) <—> (C,-); (E, -) <—> (E, +)
Funktionbezeichnung (Handbuch): T – °S; °T – $^{+}$D
Klangschlüssel (Skizze): c+ – °c; °e – e+ [MS: 13, 14]



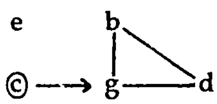
(C major)



(A minor)

II: Quintschritte

2) **schlichter Quintschritt = Q**
(C, +) —> (G, +); (E, -) —> (A, -)
Funktionbezeichnung (Handbuch): T – D; °T – °S
Klangschlüssel (Skizze): c+ – °c; °e – e+ [MS: 1, 2]



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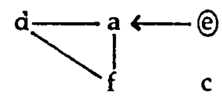
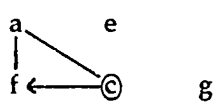


(A minor)

3) **Gegenquintschritt = -Q**
(C, +) —> (F, +); (E, -) —> (B, -)
Funktionbezeichnung (Handbuch): T – S; °T – °D
Klangschlüssel (Skizze): c+ – f+; °e – °b [MS: 7, 8]



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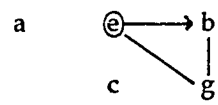
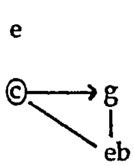


(A minor)

4) **Quintwechsel = QW**
(C, +) <—> (G, -); (E, -) <—> (A, +)
Funktionbezeichnung (Handbuch): T – (°S)D; °T – (D)°S
Klangschlüssel (Skizze): c+ – °g; °e – +a [MS: 15, 16]



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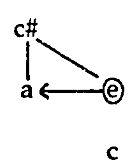
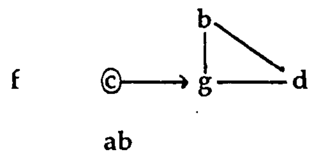
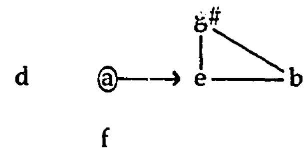


(A minor)

5) **Gegenquintwechsel = -QW**
(G, +) <—> (C, -); (A, -) <—> (E, +)
Funktionbezeichnung (Handbuch): °S – D (major and minor)
Klangschlüssel (Skizze): g+ – °c; °a – e+



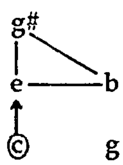
(C major)



(A minor)

III: Terzschritte

6) **schlichter Terzschritt = T**
(C, +) —> (E, +); (E, -) —> (C, -)
Funktionbezeichnung (Handbuch): T – (D)Tp, T– +3; °T – (°S)°Tp
Klangschlüssel (Skizze): c+ – e+; °e – °c [MS: 3, 4]



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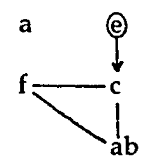
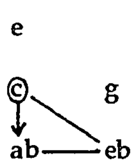
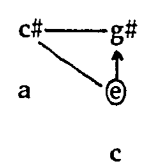


(A minor)

7) **Gegenterzschritt = -T**
(C, +) —> (Ab, +); (E, -) —> (G#, -)
Funktionbezeichnung (Handbuch): T – °Sp, T– +III; °T – +Dp
Klangschlüssel (Skizze): c+ – ab+; °e – °g# [MS: 9, 10]

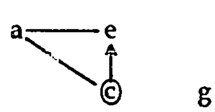


(C major)

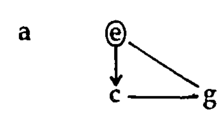


(A minor)

8) **Terzwechsel = TW**
(C, +) <—> (E, -); (E, -) <—> (C, +)
Funktionbezeichnung (Handbuch): T – Tp; °T – °Tp
Klangschlüssel (Skizze): c+ – °e+; °e – c+ [MS: 17, 18]

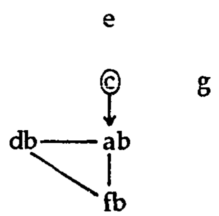


(C major)

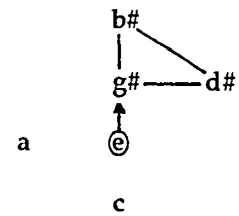


(A minor)

9) **Gegenterzwechsel = -TW**
(C, +) <—> (Ab, -); (E, -) <—> (G#, +)
Funktionbezeichnung (Handbuch): T – (°S)°Sp, T –Sv; °T – (D)+Dp
Klangschlüssel (Skizze): c+ – ab+; °e – °g#



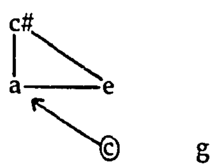
(C major)



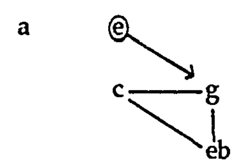
(A minor)

IV: Kleinterzschritte

10) **schlichter Kleinterzschritt = K**
(C, +) —> (A, +); (E, -) —> (G, -)
Funktionbezeichnung (Handbuch): T – (D)Sp; °T – (°S)°Dp
Klangschlüssel (Skizze): c+ – a+; °e – °g [MS: 5, 6]

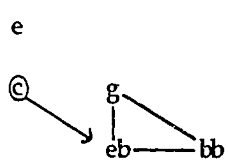


(C major)

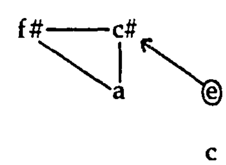


(A minor)

11) **Gegen-Kleinterzschritt = -K**
(C, +) —> (Eb, +); (E, -) —> (C#, -)
Funktionbezeichnung (Handbuch): T – (D)°Sp; °T – (°S)Dp
Klangschlüssel (Skizze): c+ – eb+; °e – °c# [MS: 11, 12]

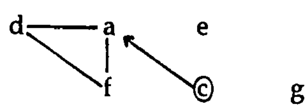


(C major)

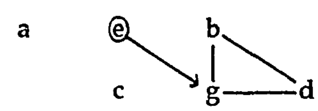


(A minor)

12) **Kleinterzwechsel = KW**
(C, +) <—> (A, -); (E, -) <—> (G, +)
Funktionbezeichnung (Handbuch): T – Sp; °T –°Dp
Klangschlüssel (Skizze): c+ – °a; °e – g+

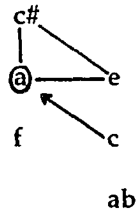


(C major)

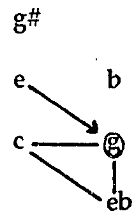


(A minor)

13) **Gegenkleinterzwechsel = -KW**
(A, +) <—> (C, -); (G, -) <—> (E, +)
Funktionbezeichnung (Handbuch): °S – (D)Sp; D – (°S)°Dp
Klangschlüssel (Skizze): °c – a+; e+ – °g



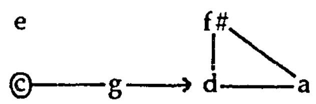
(C major)



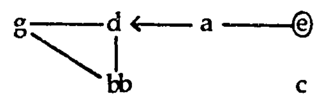
(A minor)

V: Ganztonschritte

14) **schlichter Ganztonschritt = G**
(C, +) —> (D, +); (E, -) —> (D, -)
Funktionbezeichnung (Handbuch): T – D^D; °T –$°S^S$
Klangschlüssel (Skizze): c+ – d+; °e – °d

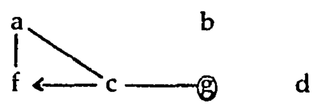


(C major)

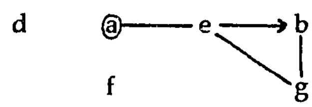


(A minor)

15) **Gegenganztonschritt = -G**
(G, +) —> (F, +); (A, -) —> (B, -)
Funktionbezeichnung (Handbuch): D – S; °S –°D
Klangschlüssel (Skizze): g+ – f+; °a – °b

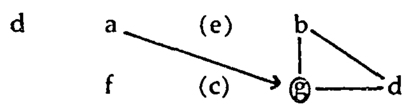


(C major)

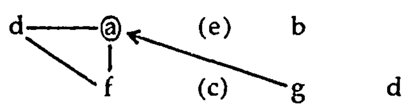


(A minor)

16) **Ganztonwechsel = GW**
(G, +) <—> (A, -); (A, -) <—> (G, +)
Funktionbezeichnung (Handbuch): Sp – D; °Dp – °S
Klangschlüssel (Skizze): °a– g+; g+– °a

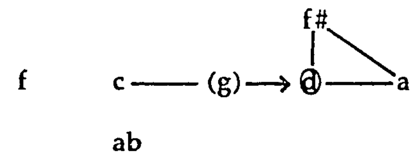


(C major)

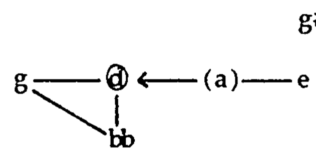


(A minor)

17) **Gegenganztonwechsel=-GW**
(D, +) <—> (C, -); (D, -) <—> (E, +)
Funktionbezeichnung (Handbuch): °S – D^D; D – °S^S
Klangschlüssel (Skizze): d+ – °c; °d – e+



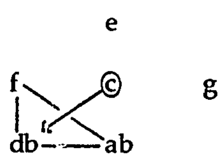
(C major)



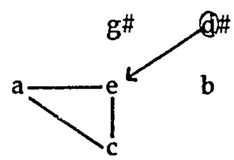
(A minor)

VI: Halbtonschritte

18) **steigender Halbtonschritt: -L, L**
(C, +) —> (Db, +); (D#, -) —> (E, -)
Funktionbezeichnung (Handbuch): T – ~~S~~; ~~D~~ – °T
Klangschlüssel (Skizze): c+ – db+ (Gegenleittonschritt) = -L
°d# – °e (schlichter Leittonschritt) = L

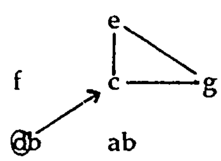


(C major)

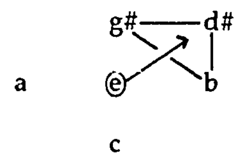


(A minor)

19) **fallender Halbtonschritt: L, -L**
(Db, +) —> (C, +); (E, -) —> (D#, -)
Funktionbezeichnung (Handbuch): ~~S~~ – T; °T – ~~D~~
Klangschlüssel (Skizze): db+ – c+ (schlichter leittonschritt) = L
°e – °d# (gegen Leittonschritt) = -L

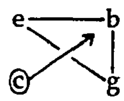


(C major)

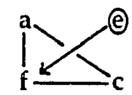


(A minor)

20) **Leittonwechsel: LW**
(C, +) <—> (B, -); (E, -) <—> (F, +)
Funktionbezeichnung (Handbuch): Dp – T, T – T̸; °Sp –°T, °T – T̸
Klangschlüssel (Skizze): c+ – °b; °e – f+ [MS: 19, 20]

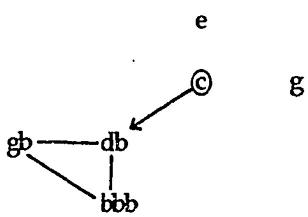


(C major)

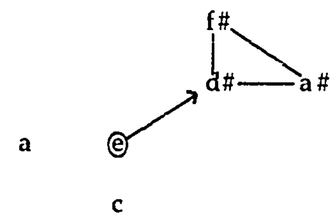


(A minor)

21) **Gegenleittonwechsel: -LW**
(C, +) <—> (Db, -); (E, -) <—> (D#, +)
Funktionbezeichnung (Handbuch): T – (°S)S̸; °T – (D)D̸
Klangschlüssel (Skizze): c+ – °db; °e – d#+



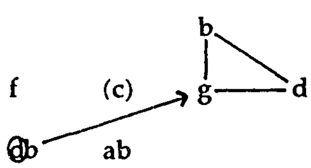
(C major)



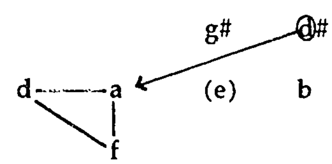
(A minor)

VII: Tritonusschritte

22) **schlichter Tritonusschritt: Z**
(Db, +) —> (G, +); (D#, -) —> (A, -)
Funktionbezeichnung (Handbuch): S̸ – D; D̸ – °S
Klangschlüssel (Skizze): db+ – g+; °d# – °a

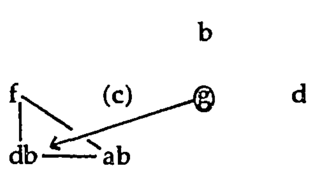


(C major)

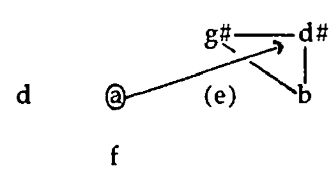


(A minor)

23) **Gegentritonusschritt: -Z**
(G, +) —> (Db, +); (A, -) —> (D#, -)
Funktionbezeichnung (Handbuch): D – S̸; °S – D̸
Klangschlüssel (Skizze): g+ – db+; °a – °d#

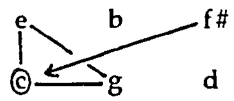


(C major)

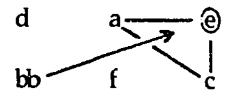


(A minor)

24) **Tritonuswechsel: ZW**
(C, +) <—> (F#, -); (E, -) <—> (Bb, +)
Funktionbezeichnung (Handbuch): D̸ – T; S̸ – °T
Klangschlüssel (Skizze): °f# – c+; °e – bb+

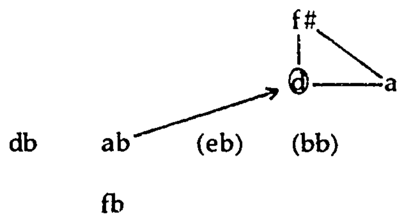


(C major)

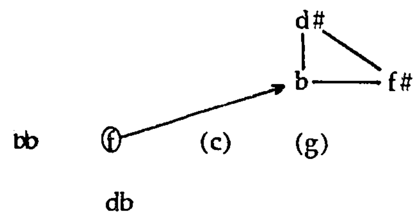


(A minor)

25) ***Gegentritonuswechsel: -ZW**
(D, +) <—> (Ab, -); (F, -) <—> (B, +)
Funktionbezeichnung (Handbuch): S̸$^{3>}$ – D^{D} (major and minor)
Klangschlüssel (not in Skizze): [°ab – d+; °f – b+]



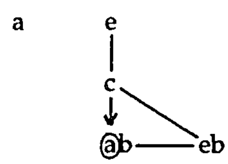
(C major)



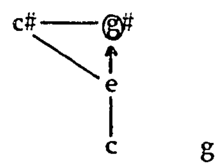
(A minor)

VIII: Other Transformations

26) **Doppelterzwechsel** = (Lewin's SLIDE)
(Ab, +) <—> (E, -); (G#, -) <—> (C, +)
Funktionbezeichnung (Handbuch): Tp – °Sp; °Tp – Dp
Klangschlüssel (Skizze): °e – ab+; c+ – °g#

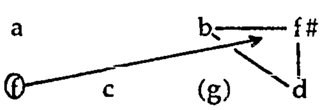


(C major)

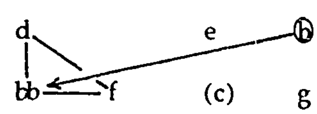


(A minor)

27) ***chromatischer Halbtonwechsel**
(F, +) <—> (F#, -); (B, -) <—> (Bb, +)
Funktionbezeichnung (Handbuch): S – D̸; °D – S̸
Klangschlüssel (not in Skizze): f+ – °f#; °b – bb+

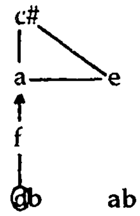


(C major)

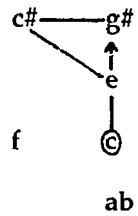


(A minor)

28) ***schlichter oder Gegen-Doppelterzschritt**
(Db, +) —> (A, +); (C, -) —> (G#, -)
Funktionbezeichnung (Handbuch): ~~S~~ – (D)Sp; (°S)Tp – ⁺Dp
Klangschlüssel (not in Skizze): db+ – a+; °c – °g#

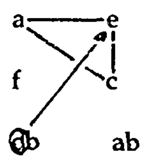


(C major)

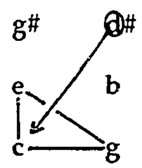


(A minor)

29) ***übermässiger Sekundwechsel**
(Db, +) <—> (E, -); (D#, -) <—> (C, +)
Funktionbezeichnung (Handbuch): ~~S~~ – Tp; ~~D~~ – °Tp
Klangschlüssel (not in Skizze): db+ – °e; °d# – c+

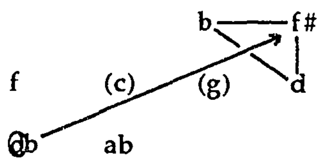


(C major)

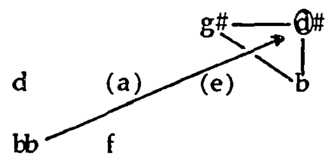


(A minor)

30) ***übermässiger Terzwechsel**
(Db, +) <—> (F#, -); (D#, -) <—> (Bb, +)
Funktionbezeichnung (Handbuch): ~~S~~ – ~~D~~ (major and minor)
Klangschlüssel (not in Skizze): db+ – °f#; bb+ – °d#

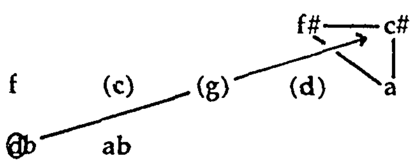


(C major)

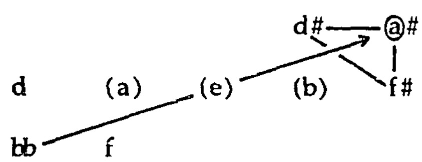


(A minor)

31) ***verminderter Gegensekundwechsel** (enharmonischer Seitenwechsel)
(Db, +) <—> (C# -); (A#, -) <—> (Bb, +)
Funktionbezeichnung (Handbuch): ~~S~~ – (~~D~~)D (major and minor)
Klangschlüssel (not in Skizze): db+ – °c#; bb+ – °a#

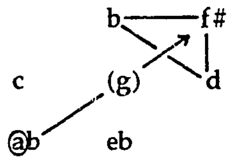


(C major)

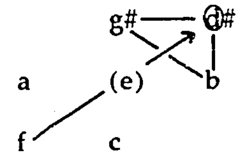


(A minor)

32) ***verminderter Kleinterzwechsel** (enharmonischer Gegenganztonwechsel)
(Ab, +) <—> (F#, -); (D#, -) <—> (F, +)
Funktionbezeichnung (Handbuch): °Sp – Đ (major and minor)
Klangschlüssel (not in Skizze): ab+ – °f#; °d# – f+



(C major)



(A minor)

APPENDIX 2: SW-SYSTEM

SPACE = {(x, sign) | x = pitch, sign = + (*Oberklang*), – (*Unterklang*), ± (*Klang*)}

Example: (C, +) = C–E–G; (C, –) = F–A^{b}–C

'+' and '–' are signs of polarity
(C, +) is the positive side of (C, ±); (C, –) is the negative side of (C, ±)
(C, ±) = C-*Klang* (F–A^{b}–C–E–G)
(C, +) = C-*Oberklang* (C–E–G)
(C, –) = C-*Unterklang* (F–A^{b}–C)

SW-system is a group of transformations on SPACE
SW-system = {I, W, {Q4}, {T4}, {L4}, {K4}, {G4}, {Z4}}

Examples:

I = Identity	(C, +) <—I—> (C, +)
W = Wechsel	(C, +) <—W—> (C, –)
Q4 = {Q, -Q, QW, -QW}	(C, +) —Q—> (G, +); (C, +) —-Q—> (F, +)
	(C, +) <—QW—> (G, –); (C, +) <—-QW—> (F, –)
T4 = {T, -T, TW, -TW}	(C, +) —T—> (E, +); (C, +) —-T—> (Ab, +)
	(C, +) <—TW—> (E, –); (C, +) <—-TW—> (Ab, –)
K4 = {K, -K, KW, -KW}	(C, +) —K—> (A, +); (C, +) —-K—> (Eb, +)
	(C, +) <—KW—> (A, –); (C, +) <—-KW—> (Eb, –)
G4 = {G, -G, GW, -GW}	(C, +) —G—> (D, +); (C, +) —-G—> (Bb, +)
	(C, +) <—GW—> (D, –); (C, +) <—-GW—> (Bb, –)
L4 = {L, -L, LW, -LW}	(C, +) —L—> (B, +); (C, +) —-L—> (Db, +)
	(C, +) <—LW—> (B, –); (C, +) <—-LW—> (Db, –)
Z4 = {Z, -Z, ZW, -ZW}	(C, +) —Z—> (F#, +); (C, +) —-Z—> (Gb, +)
	(C, +) <—ZW—> (F#, –); (C, +) <—-ZW—> (Gb, –)

The six SW-sets along with I and W comprise the SW-system on SPACE.
Appendix 1 correlates the SW-system with *Klangschlüssel*, *Funktionbezeichnungen*, and harmonic terminology from *Skizze einer neuen Methode der Harmonielehre* (2d ed., 1887), and *Handbuch der Harmonielehre* (6th ed., 1918). Riemann described all twenty-six transformations and grouped them as we have done above. Seven additional transformations described by Riemann also appear in appendix 1 (nos. 27–32). There we map all thirty-two transformations onto the Table, first in the positive (+) sense of the *Klang*, then in the negative (–) sense.
Some rules for composing transformations follow, as well as examples of noncommutativity and associativity.

Rules of Composition

1) For any (x)*Schritt* where x is of the magnitude Q, T, L, K, G, or Z, x * (-x) = I
2) For any (x)*Wechsel* where x is of the magnitude Q, T, L, K, G, or Z, x * x = I
3) (x)*Schritt* * (y)*Schritt* = (x * y)*Schritt*.
4) (x)*Wechsel* * (y)*Wechsel* = (x * –y)*Schritt*
5) (x)*Schritt* * (y)*Wechsel* = (x * y)*Wechsel*
6) (x)*Wechsel* * (y)*Schritt* = (x * –y)*Wechsel*

SW-system is a noncommutative group by virtue of rules 5 and 6.

example:

-QW * Q ≠ Q * -QW
-GW ≠ W

(F, –) —> (B^b, –) ≠ (G, +) —>(C, –)
(B^b, –) ≠ (C, –)

SW-system is associative by definition.

example:

Q * (-QW * Q) = (Q * -QW) * Q
Q * (-GW) = (W) * Q
-QW = -QW

(G, +) * (-QW * Q) = (Q * -QW) * Q
(G, +) * (-GW) = (C, –) * Q
(G, +) —> (F, –) = (C, –) —> (F, –)
(F, –) = (F, –)

APPENDIX 2 (CONT): COMPOSITION OF SW-SETS Q4, T4, K4, G4, L4, Z4

SET Q4

*	I	Q	-Q	QW	-QW
I	I	Q	-Q	QW	-QW
Q	Q	G	I	GW	W
-Q	-Q	I	-G	W	-GW
QW	QW	W	GW	I	G
-QW	-QW	-GW	W	-G	I

SET Q4

*	I	Q	-Q	QW	-QW
I	(C, +)	(G, +)	(F, +)	(G, -)	(F, -)
Q	(G, +)	(D,+)	(C, +)	(D, -)	(C, -)
-Q	(F, +)	(C, +)	(Bb, +)	(C, -)	(Bb, -)
QW	(G, -)	(C, -)	(D, -)	(C, +)	(D,+)
-QW	(F, -)	(Bb, -)	(C, -)	(Bb, +)	(C, +)

SET T4

*	I	T	-T	TW	-TW
I	I	T	-T	TW	-TW
T	T	-T	I	-TW	W
-T	-T	I	T	W	TW
TW	TW	W	-TW	I	-T
-TW	-TW	TW	W	T	I

SETT$_4$

*	I	T	-T	TW	-TW
I	(C, +)	(E, +)	(Ab, +)	(E, -)	(Ab, -)
T	(E, +)	(Ab,+)	(C, +)	(Ab, -)	(C, -)
-T	(Ab, +)	(C, +)	(Fb, +)	(C, -)	(Fb, -)
TW	(E, -)	(C, -)	(G#, -)	(C, +)	(G#,+)
-TW	(Ab, -)	(Fb, -)	(C, -)	(Fb, +)	(C, +)

SETK$_4$

*	I	K	-K	KW	-KW
I	I	K	-K	KW	-KW
K	K	Z	I	ZW	W
-K	-K	I	-Z	W	-ZW
KW	KW	W	ZW	I	Z
-KW	-KW	-ZW	W	-Z	I

SETK$_4$

*	I	K	-K	KW	-KW
I	(C, +)	(A, +)	(Eb, +)	(A, -)	(Eb, -)
K	(A, +)	(F#,+)	(C, +)	(F#, -)	(C, -)
-K	(Eb, +)	(C, +)	(Gb, +)	(C, -)	(Gb, -)
KW	(A, -)	(C, -)	(F#, -)	(C, +)	(F#,+)
-KW	(Eb, -)	(Gb, -)	(C, -)	(Gb, +)	(C, +)

SETG4

*	I	G	-G	GW	-GW
I	I	G	-G	GW	-GW
G	G	T	I	TW	W
-G	-G	I	-T	W	-TW
GW	GW	W	TW	I	T
-GW	-GW	-TW	W	-T	I

SETG4

*	I	G	-G	GW	-GW
I	(C, +)	(D, +)	(Bb, +)	(D, -)	(Bb, -)
G	(D, +)	(E, +)	(C, +)	(E, -)	(C, -)
-G	(Bb, +)	(C, +)	(Ab, +)	(C, -)	(Ab, -)
GW	(D, -)	(C, -)	(E, -)	(C, +)	(E, +)
-GW	(Bb, -)	(Ab, -)	(C, -)	(Ab, +)	(C, +)

SETL4

*	I	L	-L	LW	-LW
I	I	L	-L	LW	-LW
L	L	-G	I	-GW	W
-L	-L	I	G	W	GW
LW	LW	W	-GW	I	-G
-LW	-LW	GW	W	G	I

SET L_4

*	I	L	-L	LW	-LW
I	(C, +)	(B, +)	(Db, +)	(B, -)	(Db, -)
L	(B, +)	(A#,+)	(C, +)	(A#, -)	(C, -)
-L	(Db, +)	(C, +)	(Ebb, +)	(C, -)	(Ebb, -)
LW	(B, -)	(C, -)	(A#, -)	(C, +)	(A#,+)
-LW	(Db, -)	(Ebb, -)	(C, -)	(Ebb, +)	(C, +)

SET Z_4

*	I	Z	-Z	ZW	-ZW
I	I	Z	-Z	ZW	-ZW
Z	Z	I	I	W	W
-Z	-Z	I	I	W	W
ZW	ZW	W	W	I	I
-ZW	-ZW	W	W	I	I

SET Z_4

*	I	Z	-Z	ZW	-ZW
I	(C, +)	(F#, +)	(Gb, +)	(F#, -)	(Gb, -)
Z	(F#, +)	(B#, +)	(C, +)	(B#, -)	(C, -)
-Z	(Gb, +)	(C, +)	(Dbb,+)	(C, -)	(Dbb, -)
ZW	(F#, -)	(C, -)	(B#, -)	(C, +)	(B#, +)
-ZW	(Gb, -)	(Dbb, -)	(C, -)	(Dbb,+)	(C, +)

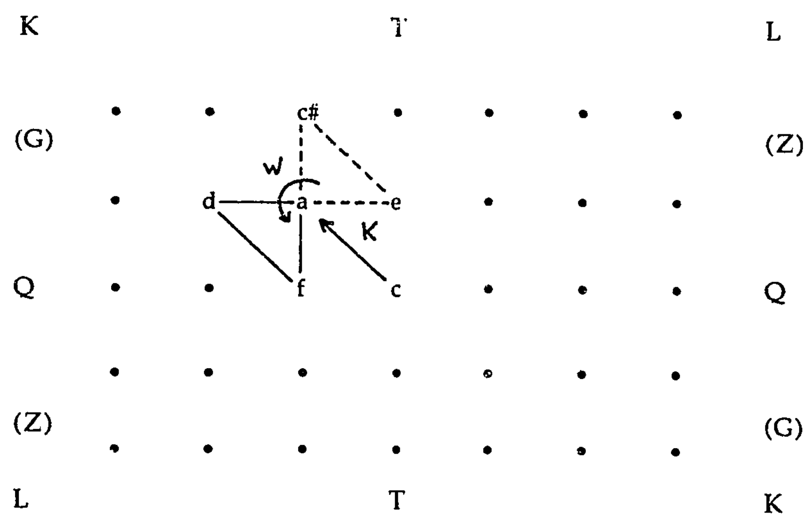
APPENDIX 2 (CONT.): PATHS TO "KW"

Mapping "KW" (Chord-of-the-added-sixth)

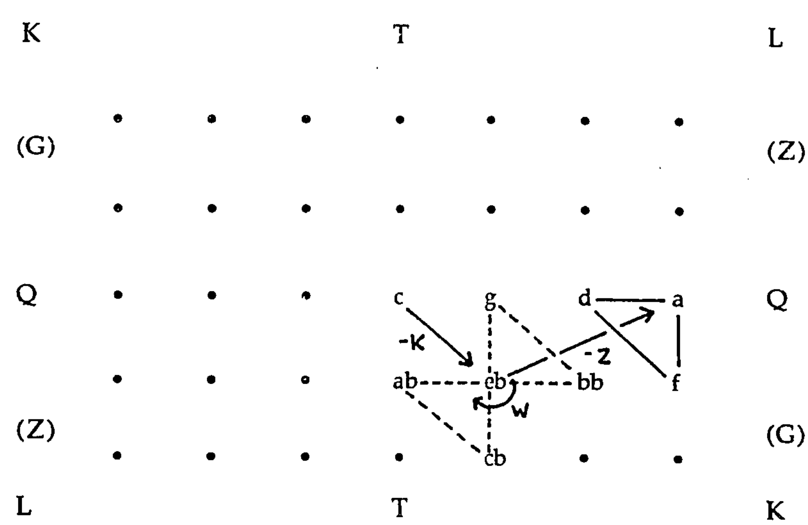
path 1: KW
path 2: (QW * T) * -Z
path 3: QW * (T * -Z)

KW	=	(QW * T) * -Z	= QW * (T * -Z)
KW	=	(-KW) * -Z	= QW * (-G)
KW	=	KW	= KW

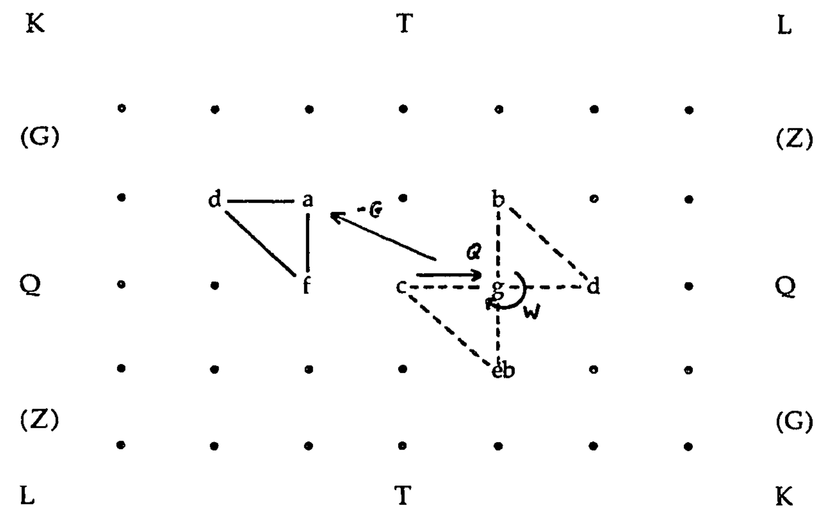
path 1:



path 2:



path 3:



APPENDIX 2A: COMPOSITION TABLE FOR SW-SYSTEM

•	I	W	Q	-Q	QW	-QW	T	-T	TW	-TW	L	-L	LW	-LW	K	-K	KW	-KW	G	-G	GW	-GW	Z	-Z	ZW	-ZW
I	I	W	Q	-Q	QW	-QW	T	-T	TW	-TW	L	-L	LW	-LW	K	-K	KW	-KW	G	-G	GW	-GW	Z	-Z	ZW	-ZW
W	W	I	-QW	QW	-Q	Q	-TW	TW	-T	T	-LW	LW	-L	L	-KW	KW	-K	K	-GW	GW	-G	G	-ZW	ZW	-Z	Z
Q	Q	QW	G	I	GW	W	L	-K	LW	-KW	Z	-T	ZW	-TW	T	-G	TW	-GW	K	-Q	KW	-QW	-L	-L	-LW	-LW
-Q	-Q	-QW	I	-G	W	-GW	K	-L	KW	-LW	T	-Z	TW	-ZW	G	-T	GW	-TW	Q	-K	QW	-KW	L	L	LW	LW
QW	QW	Q	W	GW	I	G	-KW	LW	-K	L	-TW	ZW	-T	Z	-GW	TW	-G	T	-QW	KW	-Q	K	-LW	-LW	-L	-L
-QW	-QW	-Q	-GW	W	-G	I	-LW	KW	-L	K	-ZW	TW	-Z	T	-TW	GW	-T	G	-KW	QW	-K	Q	LW	LW	L	L
T	T	TW	L	K	LW	KW	-T	I	-TW	W	-K	-Q	-KW	-QW	-L	Q	-LW	QW	Z	G	ZW	GW	-G	-G	-GW	-GW
-T	-T	-TW	-K	-L	-KW	-LW	I	T	W	TW	Q	K	QW	KW	-Q	L	-QW	LW	-G	-Z	-GW	-ZW	G	G	GW	GW
TW	TW	T	KW	LW	K	L	W	-TW	I	-T	-QW	-KW	-Q	-K	QW	-LW	Q	-L	GW	ZW	G	Z	-GW	-GW	-G	-G
-TW	-TW	-T	-LW	-KW	-L	-K	TW	W	T	I	KW	QW	K	Q	LW	-QW	L	-Q	-ZW	-GW	-Z	-G	GW	GW	G	G
K	K	KW	T	G	TW	GW	-L	-Q	-LW	-QW	-T	-G	-TW	-GW	Z	I	ZW	W	L	Q	LW	QW	-K	-K	-KW	-KW
-K	-K	-KW	-G	-T	-GW	-TW	Q	L	QW	LW	G	T	GW	TW	I	-Z	W	-ZW	-Q	-L	-QW	-LW	K	K	KW	KW
KW	KW	K	GW	TW	G	T	-QW	-LW	-Q	-L	-GW	-TW	-G	-T	W	ZW	I	Z	QW	LW	Q	L	-KW	-KW	-K	-K
-KW	-KW	-K	-TW	-GW	-T	-G	LW	QW	L	Q	TW	GW	T	G	-ZW	W	-Z	I	-LW	-QW	-L	-Q	KW	KW	K	K
G	G	GW	K	Q	KW	QW	Z	-G	ZW	-GW	-L	-K	-LW	-KW	L	-Q	LW	-QW	T	I	TW	W	-T	-T	-TW	-TW
-G	-G	-GW	-Q	-K	-QW	-KW	G	-Z	GW	-ZW	K	L	KW	LW	Q	-L	QW	-LW	I	-T	W	-TW	T	T	TW	TW
GW	GW	G	QW	KW	Q	K	-GW	-ZW	-G	-Z	-KW	-LW	-K	-L	-QW	LW	-Q	L	W	TW	I	T	-TW	-TW	-T	-T
-GW	-GW	-G	-KW	-QW	-K	-Q	ZW	GW	Z	G	LW	KW	L	K	-LW	QW	-L	Q	-TW	W	-T	I	TW	TW	T	T
L	L	LW	Z	T	ZW	TW	-K	Q	-KW	QW	-G	I	-GW	W	-T	G	-TW	GW	-L	K	-LW	KW	-Q	-Q	-QW	-QW
-L	-L	-LW	-T	-Z	-TW	-ZW	-Q	K	-QW	KW	I	G	W	GW	-G	T	-GW	TW	-K	L	-KW	LW	Q	Q	QW	QW
LW	LW	L	TW	ZW	T	Z	QW	-KW	Q	-K	W	-GW	I	-G	GW	-TW	G	-T	KW	-LW	K	-L	-QW	-QW	-Q	-Q
-LW	-LW	-L	-ZW	-TW	-Z	-T	KW	-QW	K	-Q	GW	W	G	I	TW	-GW	T	-G	LW	-KW	L	-K	QW	QW	Q	Q
Z	Z	ZW	-L	L	-LW	LW	-G	G	-GW	GW	-Q	Q	-QW	QW	-K	K	-KW	KW	-T	T	-TW	TW	I	I	W	W
-Z	-Z	ZW	-L	L	-LW	LW	-G	G	-GW	GW	-Q	Q	-QW	QW	-K	K	-KW	KW	-T	T	-TW	TW	I	I	W	W
ZW	ZW	Z	LW	-LW	L	-L	GW	-GW	G	-G	QW	-QW	Q	-Q	KW	-KW	K	-K	TW	-TW	T	-T	W	W	I	I
-ZW	-ZW	Z	LW	-LW	L	-L	GW	-GW	G	-G	QW	-QW	Q	-Q	KW	-KW	K	-K	TW	-TW	T	-T	W	W	I	I